Sign Matrices for Frames of 2^n-ons under Smith Conway and Cayley Dickson Multiplications

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Overview

Key Issues

1. Construction of the sign matrices for the frame multiplication in the 2^n-ons using Smith-Conway multiplications.
2. Show that the sign matrices in these two multiplications are Hadamard matrices.
3. Introduce Kronecker products and show that the sign matrices for the quarternions and octonions are equivalent to Kronecker products.

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Why these formulas??

There has been a great desire to develop doubling formulas that give better algebraic structures as the dimensions of the algebras so formed increase. Whenever these doubling formulas are applied, several interesting loop and algebraic properties are observed on the structures so formed.

Doubling Formulas

Cayley - Dickson Formula

The Cayley-Dickson formula is given by
$(a,b)(c,d) = (ac - \overline{d}b, da + b\overline{c})$

Smith-Conway Formula I

The Smith-Conway doubling formula is

$$(a,b)(c,d) = \begin{cases} (ac, \overline{a}d), & \text{if b=0;} \\ \left(ac - \overline{\overline{b}d}, b\overline{c} + b\left(\overline{\overline{a}.\overline{b^{-1}d}}\right)\right), & \text{if b} \neq 0. \end{cases}$$

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Doubling Formulas

Smith-Conway Formula II

There is a modified Smith-Conway doubling formula given by

$$(a,b)(c,d) = \begin{cases} (ac,\ \overline{a}d), & \text{if b=0;} \\ \left(ac - \overline{b\overline{\overline{d}}},\ \overline{\overline{\overline{b}}\overline{c}} + \overline{\overline{b}\ \overline{\overline{a}\ \overline{\overline{b^{-1}}\ \overline{d}}}}\right), & \text{if b} \neq 0; \end{cases}$$

Restriction of Formulas on the Basis

When the formula is restricted to the basis elements of the 2^n-ons, the two reduce to the form

- (a, 0)(c, 0) = (ac, 0)
- (a, 0)(0, d) = $(0, \overline{a}d)$
- (0, b)(c, 0) = $(0, \overline{\overline{b}\overline{c}})$
- (0, b)(0, d) = $(-\overline{b\overline{d}}, 0)$ = -$(d\overline{b}, 0)$

Hadamard Matrix

Hamard Matrix

A **Hadamard matrix** of degree n is a $n \times n$ matrix H with entries ± 1 such that $HH' = nI_n$. A Hadamard matrix is **normalized** if the first row and the first column consists entirely of +1's

If H is a hadamard matrix, then

1. Any two columns/rows are orthogonal of weight n
2. If some rows or columns of H are permuted, the resulting matrix is still Hadamard
3. If some rows/columns are multiplied by -1, the resulting matrix is still Hadamard
4. If H is Hadamard, H' is also Hadamard

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Sign Matrix

Sign Matrix

Let $A = \{a_1, a_2, \cdots, a_n\}$ be a multiplicative quasigroup in which $a_i \cdot a_j \in \{a_k, -a_k\}$. The sign matrix associated with A is the $n \times n$ matrix S with $S_{ij} = \begin{cases} 1, & \text{if } a_i a_j = a_k; \\ -1, & \text{if } a_i a_j = -a_k. \end{cases}$

We now consider the sign matrices of the 2^n-ons frame under the Smith-Conway multiplication

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Sign Matrix of Complex Numbers

The frame for the complex numbers is $B_{\mathbb{C}} = \{e_0 = 1, e_1 = i\}$
The sign matrix for the frame multiplication is the matrix

$$S_{\mathbb{C}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } S_{\mathbb{C}} S'_{\mathbb{C}} = 2I_2$$

Sign Matrix Of Quarternions

The frame,basis for the quarternions is
$B_{\mathbb{H}} = \{e_{00} = (e_0, 0), e_{01} = (e_1, 0), e_{10} = (0, e_0), e_{11} = (0, e_1)\}$
The sign matrix for the frame multiplication is the matrix

$$S_{\mathbb{H}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Computing $S_{\mathbb{H}} S'_{\mathbb{H}} = 4I_4$

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Computing $S_{\mathbb{H}} S'_{\mathbb{H}} = 4I_4$

Sign Matrix Of Octonions

The frame(basis) for the Octonions is $B_{\mathbb{K}} = \{e_{000} = (e_{00}, 0), e_{001} = (e_{01}, 0), e_{010} = (e_{10}, 0), e_{011} = (e_{11}, 0), e_{100} = (0, e_{00}), e_{101} = (0, e_{01}), e_{110} = (0, e_{10}), e_{111} = (0, e_{11})\}$

The sign matrix for the frame multiplication is the matrix

$$S_{\mathbb{K}} = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{array}\right)$$

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It is observed that and $S_{\mathbb{K}} S'_{\mathbb{H}} = 8I_8$

The matrix $S_{\mathbb{K}}$ can be written in the form $S_{\mathbb{K}} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$

Sign Matrix Of Sedenions

The frame,basis for the sedenions is
$B_{\mathbb{S}} = \{e_{0000} = (e_{000}, 0), e_{0001} = (e_{001}, 0), e_{0010} = (e_{010}, 0), e_{0011} = (e_{011}, 0), e_{0100} = (e_{100}, 0), e_{0101} = (e_{101}, 0), e_{0110} = (e_{110}, 0), e_{0111} = (e_{111}, 0), e_{1000} = (0, e_{000}), e_{1001} = (0, e_{001}), e_{1010} = (0, e_{010}), e_{1011} = (0, e_{011}), e_{1100} = (0, e_{100}), e_{1101} = (0, e_{101}), e_{1110} = (0, e_{110}), e_{1111} = (0, e_{111})\}$

Simply the basis elements of the Sedenions are the doubles of the octonion basis with $e_{0mnk} = (e_{mnk}, 0)$ and $e_{1mnk} = (0, e_{mnk})$ where e_{mnk} is a basis element in the octonions.

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$$S_{\mathbb{S}} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix}$$

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Sign Matrix Of Sedenions

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

Sign Matrix Of Sedenions

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

Sign Matrix Of Sedenions

$$D = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

Sign Matrix Of Sedenions

It is observed that and $S_{\mathbb{S}} S'_{\mathbb{S}} = 16 I_{16}$

It is motivating to ask;

Generalize

Is the sign matrix of the frame multiplication a Hadamard matrix for the general 2^n-ons??

ANSWER YES...

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Theorem

The sign matrix for the frame in the 2^n-ons under the Smith-Conway multiplication is a $2^n \times 2^n$ Hadamard matrix

Proof.

The sign matrix is Hadamard for $n \leq 4$ as shown above. This sets up the basis for an induction proof.

Let S, V be sign matrices for the frame multiplication in the 2^n-ons and 2^{n+1}-ons respectively

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Proof (Cont.)

By induction hypothesis, we assume S is Hadamard, and show V is Hadamard.

Now, $V = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$

Where A, B, C and D are $2^n \times 2^n$ matrices.

Let $\{e_0,\ e_1,\ \cdots,\ e_{2^n-1}\}$ be the basis elements of the 2^n-ons. Then the basis of the 2^{n+1}-ons is

$(e_0, 0),\ (e_1, 0), \cdots,\ (e_{2^n-1}, 0),\ (0, e_0),\ (0, e_1), \cdots,\ (0, e_{2^n-1})$

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□

Proof (Cont.)

$A_{ij} = \text{sign}[(e_{i-1}, 0)(e_{j-1}, 0)] = \text{sign}(e_{i-1} \cdot e_{j-1}, 0) =$
$\text{sign}(e_{i-1} \cdot e_{j-1}) = S_{ij}$
In this case $A = S$.

$B_{ij} = \text{sign}[(e_{i-1}, 0)(0, e_{j-1})] = \text{sign}(0, \overline{e_{i-1}} \cdot e_{j-1}) =$
$\text{sign}(\overline{e_{i-1}} \cdot e_{j-1})$
Thus $B_{1j} = \text{sign}(e_{j-1}) = 1$
For $i \neq 1$, $B_{1j} = \text{sign}(-e_{i-1} \cdot e_{j-1}) = -S_{ij}$

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Proof (Cont.)

$C_{ij} = \text{sign}[(0, e_{i-1})(e_{j-1}, 0)] = \text{sign}(0, e_{j-1} \cdot e_{i-1}, 0) = \text{sign}(e_{j-1} \cdot e_{i-1}) = S_{ji}$

In this case $C = S'$.

$D_{ij} = \text{sign}[(0, e_{i-1})(0, e_{j-1})] = \text{sign}(-e_{j-1} \cdot \overline{e_{i-1}}, 0) = \text{sign}(-e_{j-1} \cdot \overline{e_{i-1}})$

Thus $D_{1j} = \text{sign}(-e_{j-1}) = -1$

For $i \neq 1$, $D_{1j} = \text{sign}(e_{j-1} \cdot e_{i-1}) = S_{ji}$

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□

Proof (Cont.)

$$A = \left(\begin{array}{c|cccccc} 1 & 1 & 1 & . & . & . & 1 & 1 \\ \hline 1 & & & & & & & \\ 1 & & & & & & & \\ . & & & & & & & \\ . & & & & S_{ij} & & & \\ . & & & & & & & \\ & & & & & & & \\ 1 & & & & & & & \end{array}\right)$$

□

Proof (Cont.)

$$B = \left(\begin{array}{c|ccccccc} 1 & 1 & 1 & . & & . & & . & 1 & 1 \\ \hline -1 & & & & & & & & & \\ -1 & & & & & & & & & \\ . & & & & & & & & & \\ . & & & & & -S_{ij} & & & & \\ . & & & & & & & & & \\ & & & & & & & & & \\ -1 & & & & & & & & & \end{array}\right)$$

□

Proof (Cont.)

$$C = \left(\begin{array}{c|cccccc} 1 & 1 & 1 & . & . & . & 1 & 1 \\ \hline 1 & & & & & & & \\ 1 & & & & & & & \\ . & & & & & & & \\ . & & & & S_{ji} & & & \\ . & & & & & & & \\ & & & & & & & \\ 1 & & & & & & & \end{array}\right) = S'$$

□

Proof (Cont.)

$$D = \left(\begin{array}{c|cccccc} -1 & -1 & -1 & . & . & . & -1 & -1 \\ \hline 1 & & & & & & & \\ 1 & & & & & & & \\ . & & & & & & & \\ . & & & & S_{ji} & & & \\ . & & & & & & & \\ & & & & & & & \\ 1 & & & & & & & \end{array}\right)$$

□

Proof (Cont.)

Combining these we have

$$V = \left(\begin{array}{ccccc|ccccc} 1 & . & . & . & 1 & 1 & . & . & . & 1 \\ . & & & & & -1 & & & & \\ . & & S_{ij} & & & . & & -S_{ij} & & \\ . & & & & & . & & & & \\ 1 & & & & -1 & & & & & \\ \hline 1 & . & . & . & 1 & -1 & . & . & . & -1 \\ . & & & & & 1 & & & & \\ . & & S_{ji} & & & . & & S_{ji} & & \\ . & & & & & . & & & & \\ 1 & & & & & 1 & & & & \end{array}\right)$$

□

Proof (Cont.)

We perform a row permutation $R_1 \leftrightarrow R_{2^n+1}$ to get an equivalent matrix

$$H = \left(\begin{array}{ccccc|ccccc} 1 & . & . & . & 1 & -1 & . & . & . & -1 \\ . & & & & & -1 & & & & \\ . & & S_{ij} & & & . & & -S_{ij} & & \\ . & & & & & . & & & & \\ 1 & & & & -1 & & & & & \\ \hline 1 & . & . & . & 1 & 1 & . & . & . & 1 \\ . & & & & & 1 & & & & \\ . & & S_{ji} & & & . & & S_{ji} & & \\ . & & & & & . & & & & \\ 1 & & & & & 1 & & & & \end{array}\right) = \left(\begin{array}{c|c} S & -S \\ \hline S' & S' \end{array}\right)$$

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$$\left(\begin{array}{c|c} S & -S \\ \hline S' & S' \end{array}\right)$$

Proof (Cont.)

Now, $HH' = \left(\begin{array}{c|c} S & -S \\ \hline S' & S' \end{array}\right)\left(\begin{array}{c|c} S' & S \\ \hline -S' & S \end{array}\right)$

$$= \left(\begin{array}{c|c} SS' + SS' & S^2 - S^2 \\ \hline (S')^2 - (S')^2 & S'S + S'S \end{array}\right)$$

$$= \left(\begin{array}{c|c} 2^{n+1} I_{2^n} S & 0 \\ \hline 0 & 2^{n+1} I_{2^n} \end{array}\right) = 2^{n+1} I_{2^{n+1}}$$

H is Hadamard equivalent to V, and the proof is complete. □

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Sign Matrices under Cayley-Dickson Formula

The sign matrices for the frame multiplication under the Cayley-Dickson formula is a Hadamard matrix.

The sign matrix for the complex numbers is exactly the same as under Smith-Conway formula.

The matrices are different for quartenions, octonions and the general 2^n-ons.
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Example, the sign matrix for the quartenion frame multiplication under Cayley-Dickson formula is

$$S_{\mathbb{H}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

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The sign matrix under Cayley-Dickson can be achieved from the Smith-Conway matrix via the permutations $C_3 \leftrightarrow C_4$ followed by $R_3 \leftrightarrow R_4$.

The general proof that the sign matrix for the 2^n-on is Hadamard is similar to the one above except that the column permutation $C_1 \leftrightarrow C_{2^n+1}$ to get an equivalent matrix instead of a row permutation $R_1 \leftrightarrow R_{2^n+1}$.

Also the matrix for the 2^{n+1}-ons frame after the permutation is of the form $H = \left(\begin{array}{c|c} S & S' \\ \hline -S & S' \end{array}\right)$ instead of $H = \left(\begin{array}{c|c} S & -S \\ \hline S' & S' \end{array}\right)$

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Skew Hadamard Matrices

Definition

An Hadamard matrix of order m is **skew** if $H = S + I$ and $S' = -S$

If $H = S + I$ is a skew Hadamard matrix, the matrix S can be written in the form $S = \begin{pmatrix} 0 & e \\ -e' & W \end{pmatrix}$
where e a all ones row vector. The matrix W is called the **kernel** of the skew Hadamard matrix

Theorem

Skew Matrix Property

The sign matrices of the 2^n-ons under the Smith-Conway or Cayley-Dickson multiplication is equivalent to a skew Hadamard matrix.

Proof.

The sign matrix of the frame under these two multiplications is of the form

$$\widetilde{H} = \begin{pmatrix} 1 & & . & . & . & 1 \\ 1 & -1 & & \widetilde{H}_{ij} & & \\ . & & . & & & . \\ . & & & . & & . \\ . & \widetilde{H}_{ji} & & & & \\ 1 & . & . & . & . & -1 \end{pmatrix} \text{ with } \widetilde{H}_{ji} = -\widetilde{H}_{ij}$$

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□

Proof (Cont.)

If we multiply all rows by -1 except the first one, we get an equivalent matrix

$$H = \begin{pmatrix} 1 & & . & . & . & 1 \\ -1 & 1 & & H_{ij} & & \\ . & & . & & & . \\ . & & & . & & . \\ . & H_{ji} & & & & \\ -1 & . & . & . & . & 1 \end{pmatrix}$$

$$H = I + \begin{pmatrix} 0 & 1 & . & . & . & 1 \\ -1 & 0 & & H_{ij} & & \\ . & & . & & & . \\ . & & & . & & . \\ . & H_{ji} & & & & \\ -1 & . & . & . & . & 0 \end{pmatrix} = I + S$$

where $H_{ji} = -H_{ij}$

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Proof (Cont.)

$$S' = \begin{pmatrix} 0 & -1 & . & . & . & -1 \\ 1 & 0 & & H_{ji} & & \\ . & & . & & & . \\ . & & & . & & . \\ . & H_{ji} & & & & \\ 1 & . & . & . & . & 0 \end{pmatrix} = -S$$

and $S = \begin{pmatrix} 0 & e \\ -e' & W \end{pmatrix}$

W is the kernel of the sign matrix.

□

Kronecker Products

Definition

Let $A = [a_{ij}]$ be a $n \times m$ matrix and let $B = [b_{ij}]$ be a $r \times s$ matrix. The **Kronecker product** of A and B is the $nk \times ms$ matrix given by

$$A \otimes B == \begin{pmatrix} a_{11}B & . & . & . & . & a_{1m}B \\ . & & . & & & \\ . & & & . & & . \\ . & & & . & & . \\ a_{n1}B & . & . & . & . & a_{nm}B \end{pmatrix}$$

The Kronecker product is also known as the **direct product** or the **tensor products**.

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Properties

Kronecker products satisfy the following properties

1. $A \otimes (B + C) = A \otimes B + A \otimes C$ and $(A + B) \otimes C = A \otimes C + B \otimes C$
2. $\lambda(A \otimes B) = (\lambda A) \otimes B = A \otimes (\lambda B)$
3. $\lambda A \otimes \beta B = \lambda\beta(A \otimes B)$
4. The product is associative $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
5. If A, B, C, D are square matrices such that AC and BD exist, then $(A \otimes B)(C \otimes D) = AC \otimes BB$
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Kronecker Products of Hadamard Matrices

If H and G are Hadamard matrices of order n and m, the Kronecker product $H \times G$ is a Hadamrd matrix

Proof.

$(H \times G)(H \times G)' = (H \times G)(H' \times G') = HH' \times GG' = nI_n \otimes mI_m = nmI_{nm}$

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Kronecker Products of Sign Matrices

The Kronecker product of the sign matrix for the complex numbers is

$$S_{\mathbb{C}} \otimes S_{\mathbb{C}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

This matrix is equivalent to the sign matrix

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Kronecker Products of Sign Matrices

This matrix is mimilar to the sign matrix of the octonions via the permutation $R_2 \to R_7 \to R_3 \to R_8 \to R_4 \to R_6 \to R_2$

If we start with the Kronecker product $S_{\mathbb{H}} \otimes S_{\mathbb{C}}$, we get the equivalent matrix to $S_{\mathbb{K}}$. The row permutations $R_2 \to R_6 \to R_4 \to R_2, \; R_7 \to R_8 \to R_7$

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Question

Is the Kronecker product of the sign matrices of the 2^{n-1}-ons with that of $S_{\mathbb{C}}$ always equivalent to the sign matrix of 2^n?? If such an equivalence exists, what is the permutation matrix for equivalence??

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The problem would be solved if these two matrices H and $S_{\mathbb{C}} \otimes R$ were shown to be equivalent.

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Thank You for Listening