Algebras with operators, Koszul duality, and conformal algebras

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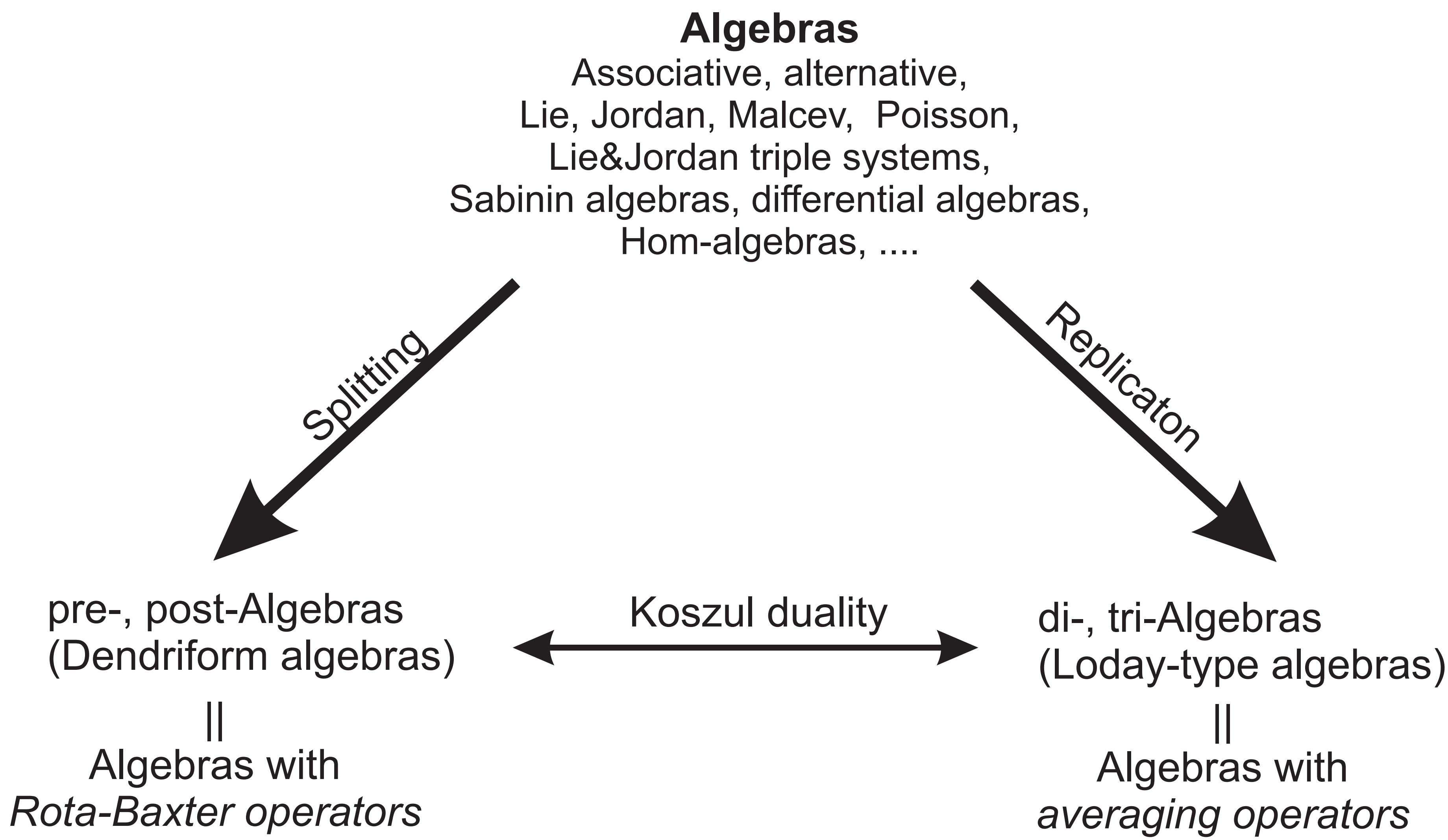
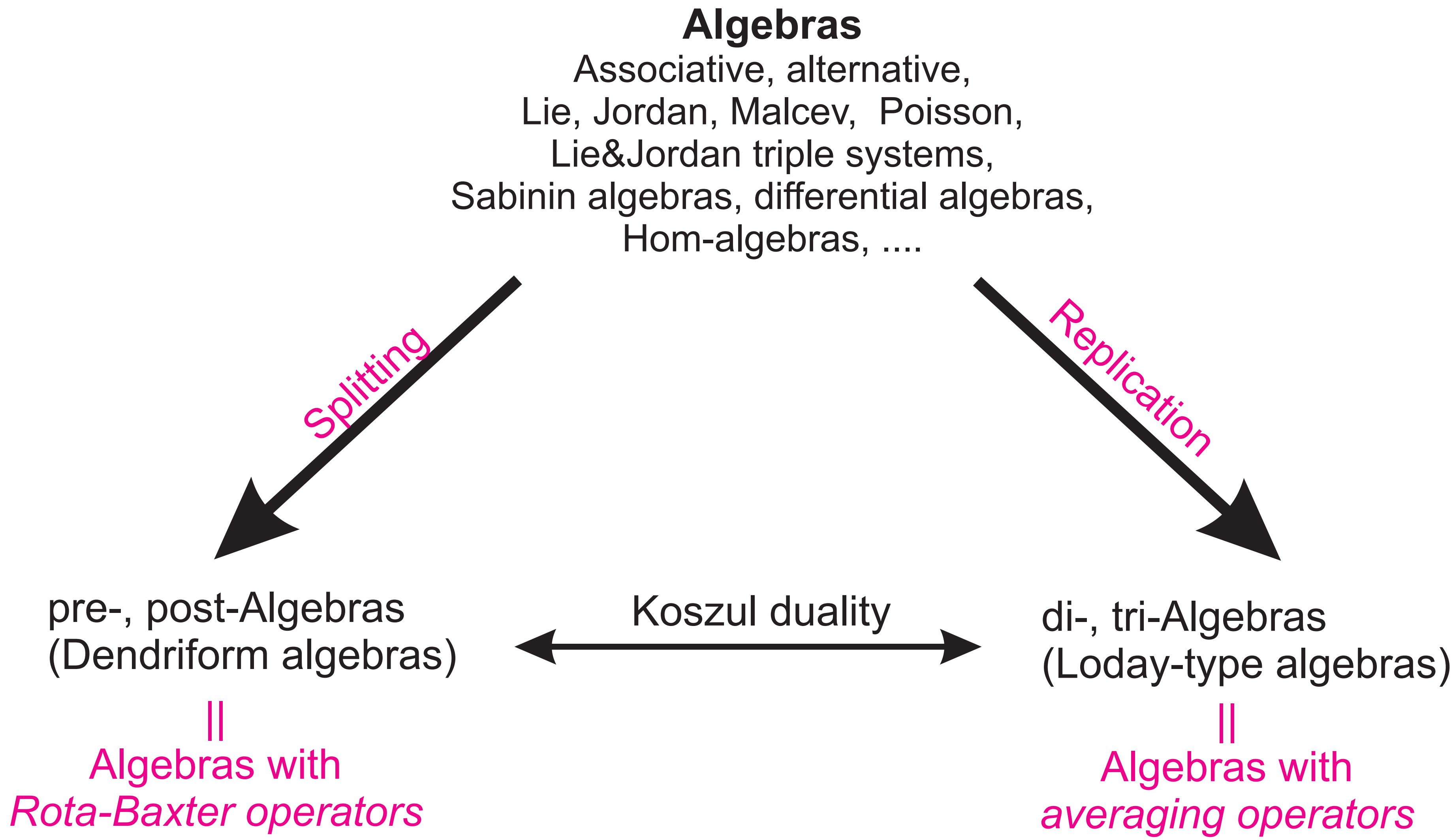

Algebras
Associative, alternative,
Lie, Jordan, Malcev, Poisson,
Lie&Jordan triple systems,
Sabinin algebras, differential algebras,
Hom-algebras, ...
Splitting
Replicaton
pre-, post-Algebras
(Dendriform algebras)
||
Algebras with
Rota-Baxter operators
Koszul duality
di-, tri-Algebras
(Loday-type algebras)
||
Algebras with
averaging operators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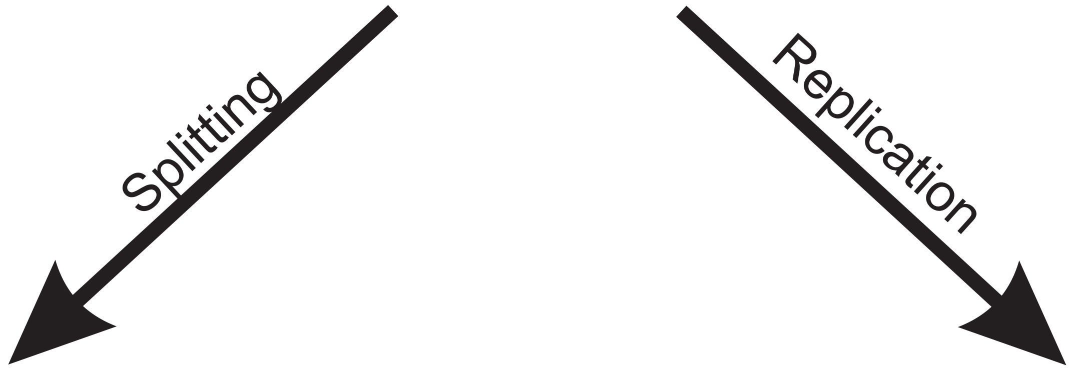
Example:

Lie Algebras

$$xy + yx = 0$$

$$(xy)z + (yz)x + (zx)y = 0$$



Left-symmetric algebras
(pre-Lie algebras)

$$(xy)z - x(yz) = (yx)z - y(xz)$$

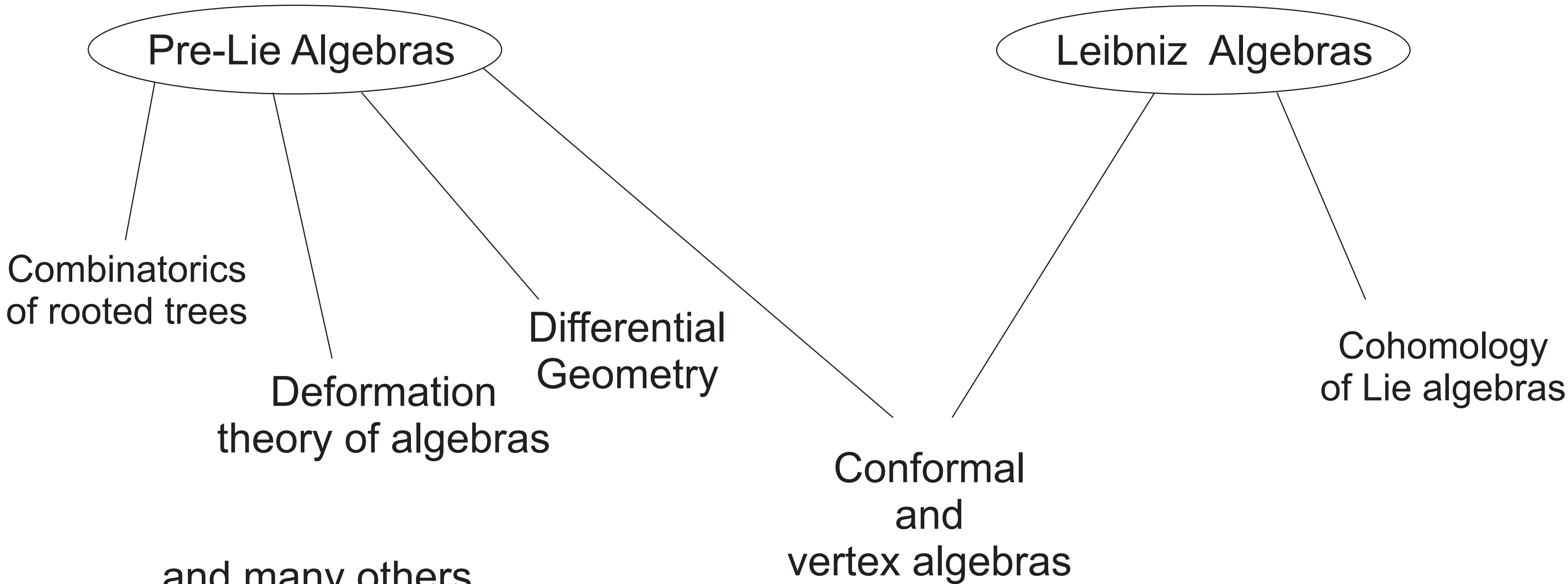
[E. Vinberg, 1960]
[J.-L. Koszul, 1961]
[M. Gerstenhaber, 1963]

[A. Cayley, 1896]

Leibniz Algebras

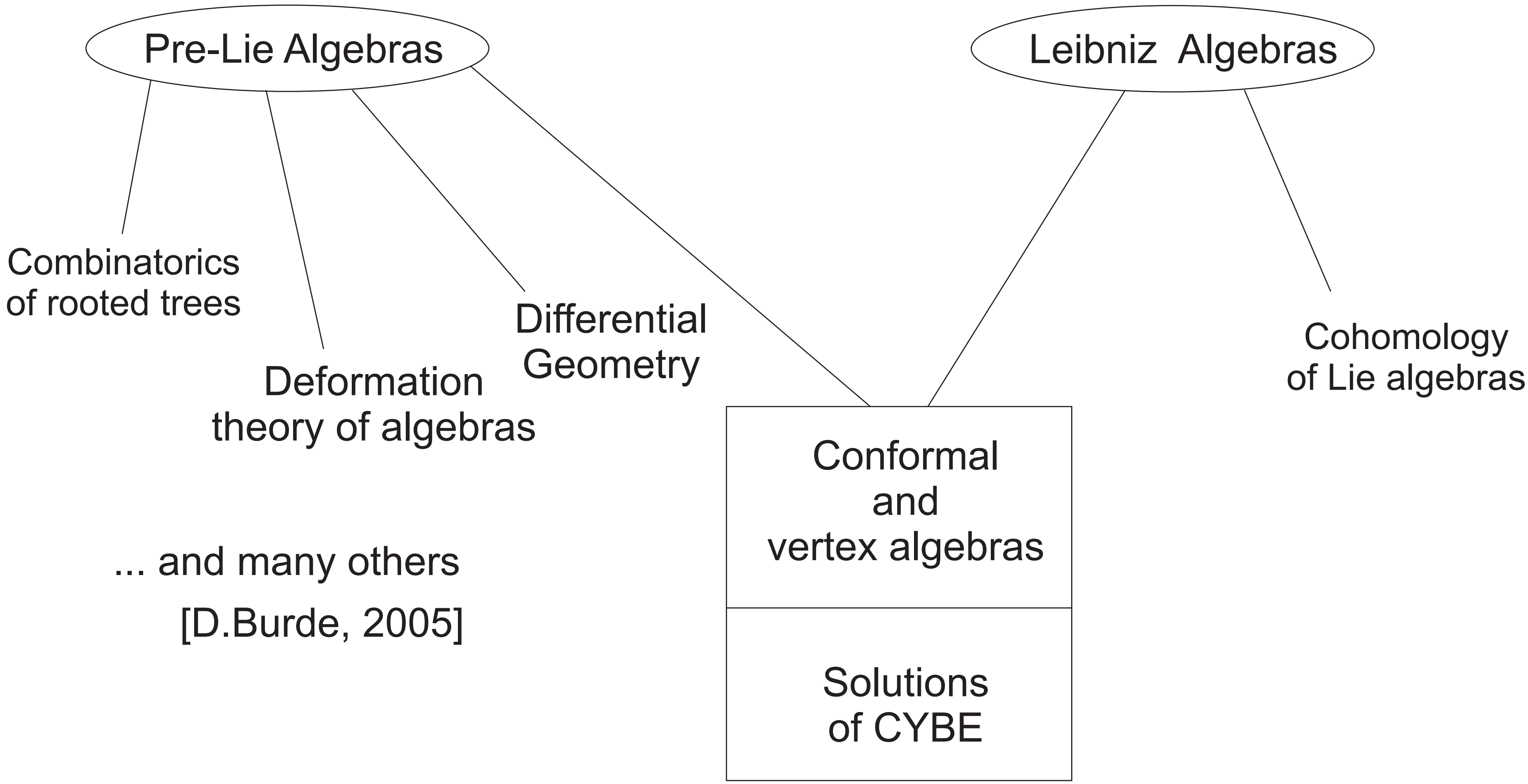
$$x(yz) = (xy)z + y(xz)$$

[A. Bloh, 1965]
[J.-L. Loday, 1993]



... and many others

[D.Burde, 2005]


Pre-Lie Algebras
Leibniz Algebras
Combinatorics of rooted trees
Deformation theory of algebras
Differential Geometry
Conformal and vertex algebras
Solutions of CYBE
Cohomology of Lie algebras
... and many others
[D.Burde, 2005]

2d-Conformal field theory

V space of states

$T : V \to V$ translation operator

$Y : V \to \mathrm{gl}\, V[[z, z^{-1}]]$ state-field correspondence

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$$

+ axioms [R. Borcherds, 1986], [V. Kac, 1996]

Locality: $[Y(a, z), Y(b, w)](z - w)^N = 0$ for $N \gg 0$

T-invariance: $[T, Y(a, z)] = \partial_z Y(a, z)$

$Y(a, z)Y(b, z) =$ undefined

$Y(a, z)Y(b, w)$ has a singularity in $z = w$

Operator product expansion formula:

$$Y(a,z)Y(b,w) = \sum_{n=0}^{N(a,b)} \frac{Y(c_n, w)}{(z-w)^{n+1}} + \langle\text{principal part}\rangle$$

$c_n = [a_{(n)}b]$, $n \geq 0$:

Algebraic operations on V

$(V, T, [\cdot_{(n)}\cdot], n \in \mathbb{Z})$

Conformal algebra

(Vertex Lie algebra)

$(V, [\cdot_{(0)}\cdot])$ Leibniz algebra

$= Y(:ab:, w) + (z-w)(\dots)$

Wick product

$(V, :\cdot\cdot:)$ pre-Lie algebra

[M. Rosellen, 2005]

Classical Yang-Baxter equation (dynamic)

$$X_{13}(u+v)X_{23}(v) - X_{23}(v)X_{12}(u) - X_{21}(u)X_{13}(u+v) = 0$$

A Algebra

$X : \mathcal{D} \to A \otimes A$ meromorphic function

$\mathcal{D} \subset \mathbb{C}$

$X_{23} = 1 \otimes X$

$X_{21} = X^{(12)} \otimes 1$, etc.

If A has nondegenerate symmetric associative bilinear form $\langle \cdot, \cdot \rangle$ then

$$X(u) \in A \otimes A \quad \leftrightarrow \quad P_u(\cdot) : A \to A$$

$$X = \sum x_i \otimes y_i \quad \leftrightarrow \quad P_u(x) = \sum \langle x_i, x \rangle y_i$$

CYBE (operator form):

$$P_{u+v}(x)P_v(y) = P_v(P_u(x)y) + P_{u+v}(xP_u(y))$$

For Lie algebras [A. Belavin, V. Dinfeld, 1982]

$$[P_{u+v}(x), P_v(y)] = P_v([P_u(x), y]) + P_{u+v}([x, P_u(y)])$$

$X \equiv$ const:

$[P(x), P(y)] = P([P(x), y] + [x, P(y)])$ $P : L \to L$ is a Baxter operator

[M. Semenov-Tian-Shansky, 1983]

$x * y = [P(x), y]$, $x, y \in L$: left-symmetric product

$(L, *)$ pre-Lie algebra

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Define $x_\lambda = \operatorname{Res}_{u=0} e^{\lambda u} P_u(x) \in L[\lambda]$

Then

$$[x_\lambda, y_\mu] = [x_\lambda, y]_{\lambda+\mu}$$

Conformal Leibniz identity for $[x_{(n)} y]$ given by $[x_\lambda, y] = \sum_{n \geq 0} \frac{\lambda^n}{n!} [x_{(n)} y]$

Replicating an operad

Perm:

$$x(yz) = (xy)z$$
$$(xy)z = (yx)z$$

$\dim \mathrm{Perm}(n) = n$:

$$e_i^{(n)} = (x_1 \overset{i}{\ldots\hat{}\ldots} x_n) x_i$$

ComTrias: Perm &

$$x \perp y = y \perp x$$
$$(x \perp y) \perp z = x \perp (y \perp z)$$
$$(x \perp y)z = (xy)z$$
$$x(y \perp z) = (xy) \perp z$$

$\dim \mathrm{ComTrias}(n) = 2^n - 1$:

$$e_{i1,\ldots,ik}^{(n)} = (x_1 \overset{i1}{\ldots\hat{}\ldots} \cdots \overset{ik}{\ldots\hat{}\ldots} x_n)(x_{i1} \perp \cdots \perp x_{ik})$$

Example:

$(C, \cdot)$ commutative algebra over a field $\Bbbk$

$\varepsilon : C \to \Bbbk$ homomorphism
e.g., counit of a bialgebra

Then
$xy = \varepsilon(x) \cdot y$
$x \perp y = x \cdot y$
turns C into a ComTrias-algebra

$C = \Bbbk\Gamma$ group algebra
$\varepsilon(g) = 1,\ g \in \Gamma$

Replicating an operad

[F. Chapoton, 2001]: $\mathrm{Leib} = \mathrm{Lie} \circ \mathrm{Perm}$ (Manin white product of operads [V. Ginzburg, M. Kapranov, 1994])

[B. Vallette, 2008]: $\mathcal{P} \circ \mathrm{Perm} = \mathcal{P} \otimes \mathrm{Perm}$ (Hadamard product of operads)

$$\mathcal{P} \circ \mathrm{ComTrias} = \mathcal{P} \otimes \mathrm{ComTrias}$$

for binary quadratic operads

Definition:

$$\text{di-}\mathcal{P} = \mathcal{P} \otimes \mathrm{Perm} \qquad \text{tri-}\mathcal{P} = \mathcal{P} \otimes \mathrm{ComTrias}$$

[P.K., 2008]
[A. Pozhidaev, 2009] } binary case (di-)

[M. Bremner, R. Felipe, J. Sanchez-Ortega, 2011] - general case (di-)
[V. Gubarev, P.K., 2011] - binary case (tri-)

[J.Pei, C. Bai, L. Guo, X. Ni, 2012] - name of the procedure

Replicating an operad
(general tri-algebra case)

$\Omega = \{f_1, f_2, \dots\}$
$\nu(f_k) = n_k \geq 1$

$\Omega_3 = \{f_1^H, f_2^H, \dots\}$
$\nu(f_k^H) = n_k \geq 1$
$H \subseteq \{1, \dots, n_k\},\ H \neq \varnothing$

$\mathcal{P}$-Algebra:
(A, Ω^A)
$f_k^A : \underbrace{A \otimes \cdots \otimes A}_{n_k} \to A$

$\Longrightarrow$

tri-$\mathcal{P}$-Algebra:
(D, Ω_3^D)

Defining identities Σ (multi-linear)

Defining identities Σ_3 (multi-linear)

$$C = \mathrm{ComTrias}\langle x_1, x_2, \dots \rangle$$

$$\boxed{A \models \Sigma \quad \Longleftrightarrow \quad C \otimes A \models \Sigma_3}$$

$$f^H(c_1 \otimes a_1, \dots, c_n \otimes a_n) = e_H^{(n)}(c_1, \dots, c_n) \otimes f(a_1, \dots, a_n)$$

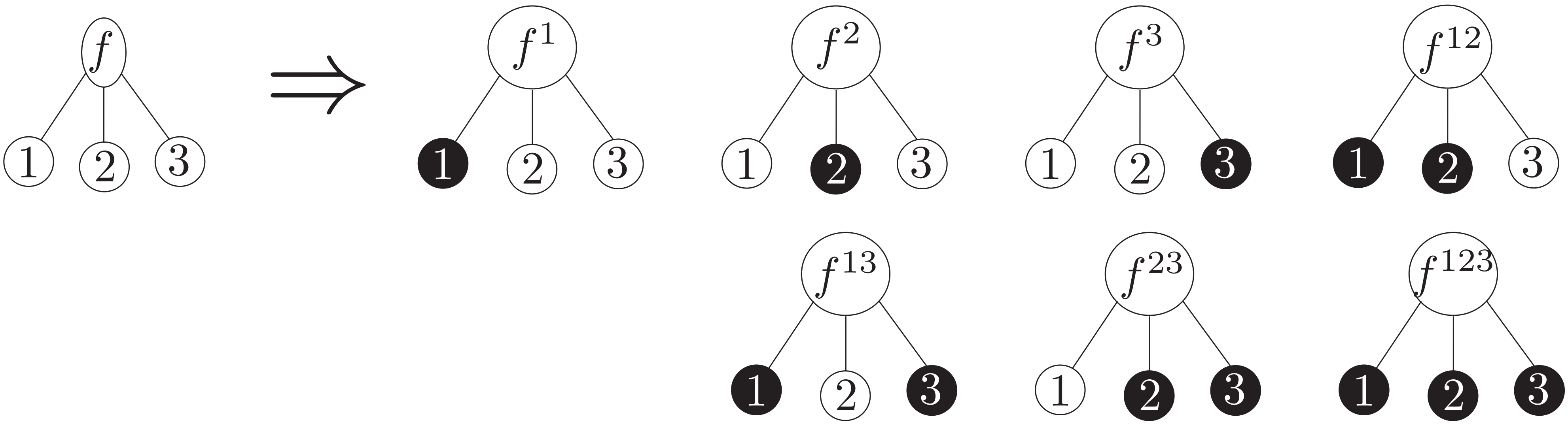
Identities Σ_3 may be derived explicitly from Σ

Bremner & Sanchez-Ortega + Gubarev & K.
(Perm instead of ComTrias)

Replicating an operad

Example: replicating a ternary operation

$f(x_1, x_2, x_3)$



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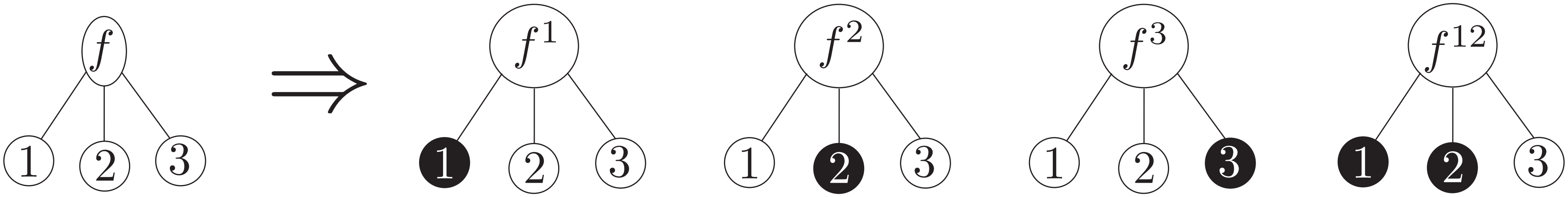
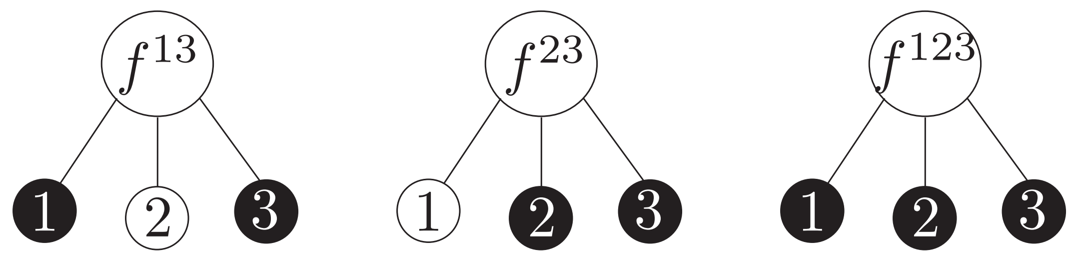


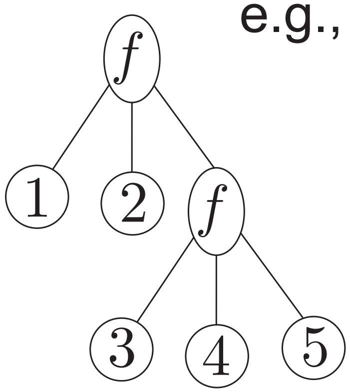


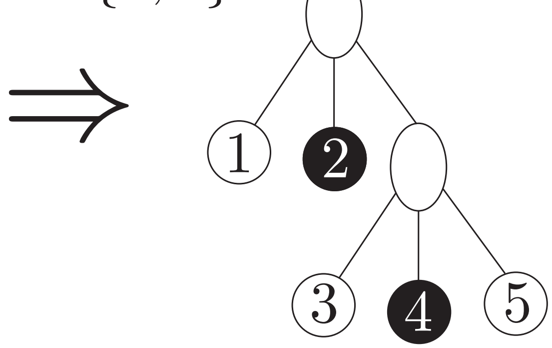
Replicating a monomial:

$f(x_1, x_2, f(x_3, x_4, x_5))$

$H \subseteq \{1, 2, 3, 4, 5\}$

e.g., $H = \{2, 4\}$:





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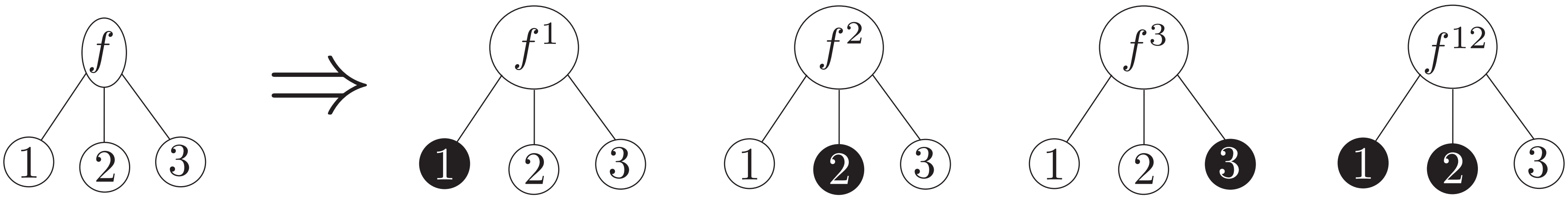

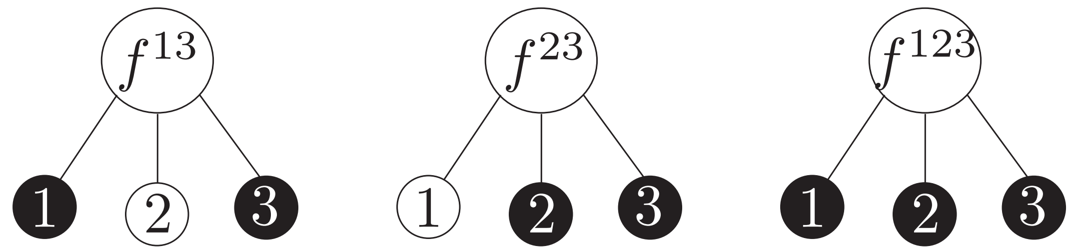

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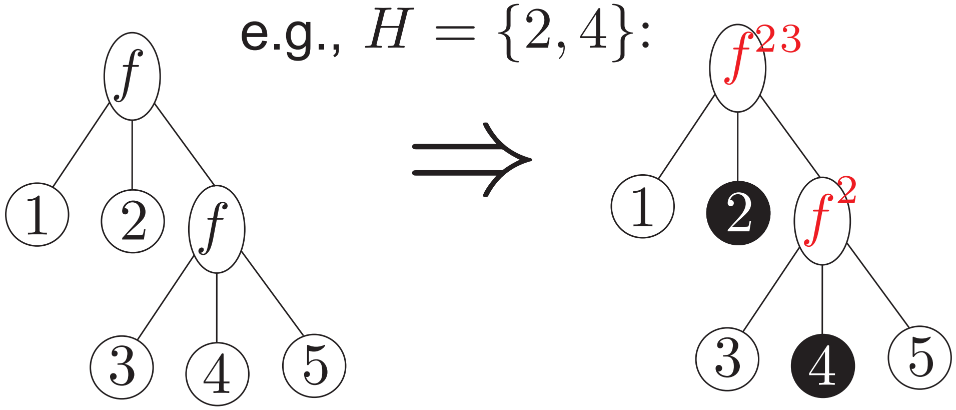

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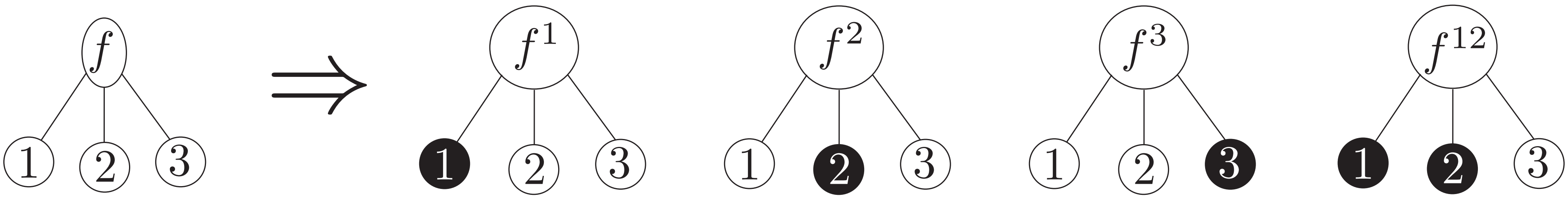

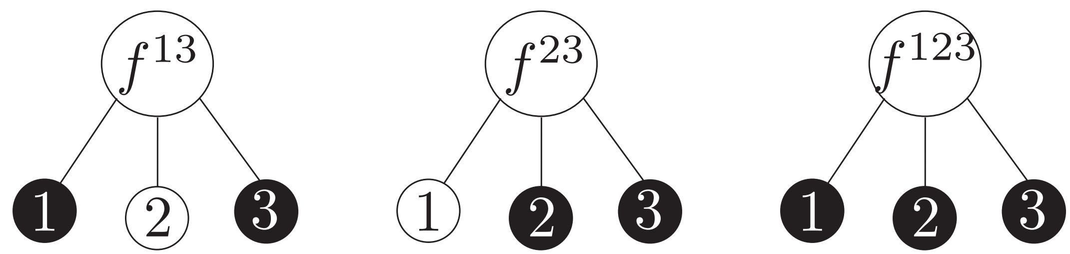

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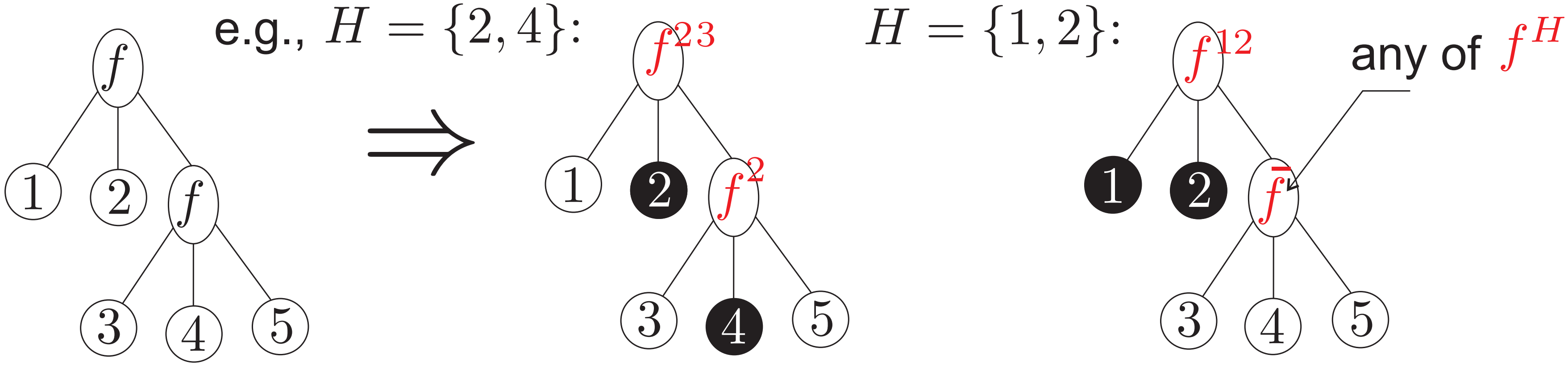

Replicating an operad

Examples

di-Lie = Leib

$$\mu(x,y) = -\mu(y,x) \Rightarrow \boxed{\mu^1(x,y) = -\mu^2(y,x)}$$

$$xy = \mu^2(x,y) \Rightarrow \text{Left Leibniz algebra}$$

$$xy = \mu^1(x,y) \Rightarrow \text{Right Leibniz algebra}$$

di-Com = Perm [F. Chapoton, 2001]
di-As = Dias [J.-L. Loday, T. Pirashvili, 1993]
di-Alt [L. Dong, 2005]
di-Jord [M. Bremner, L. Peresi, 2011]
di-JTS [M. Bremner, R. Felipe, J. Sanchez-Ortega, 2011]
di-Mal [M. Bremner, L. Peresi, J. Sanchez-Ortega, 2011]
tri-Com = ComTrias [B. Vallette, 2007]
tri-As = Trias [J.-L. Loday, M. Ronco, 2004]

Replicating an operad

Theorem (c.f. [P.K., 2008], [A. Pozhidaev, 2009], [P.K., V. Voronin, 2013])
Let $\nu(f) \geq 2$ for all $f \in \Omega$.
Then for every (di-)tri-$\mathcal{P}$-algebra D there exists canonical $\mathcal{P}$-algebra $\widehat{D}$ and a (Perm-) ComTrias-algebra C such that

$$D \subset C \otimes \widehat{D}$$

Remark.
Theorem remains valid for algebras with derivations or endomorphisms in the language.

Corollary.
$\dim \text{di-}\mathcal{P}(n) = n \dim \mathcal{P}(n)$
$\dim \text{tri-}\mathcal{P}(n) = (2^n - 1) \dim \mathcal{P}(n)$

Replicating an operad
(Averaging operators)

Demo case:
one binary operation
$\mu(x, y) = xy$

Definition:
$T : A \to A$ *averaging operator*
if $T(x)T(y) = T(T(x)y) = T(xT(y))$
for $x, y \in A$

Corollary. If A is a $\mathcal{P}$-algebra with an averaging operator T then A w.r.t.

$$\mu^1(x, y) = xT(y), \quad \mu^2(x, y) = T(x)y$$

is a di-$\mathcal{P}$-algebra.

Corollary. For every di-$\mathcal{P}$-algebra D there exists a $\mathcal{P}$-algebra A with an averaging operator T such that D embeds into A.

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If, in addition,
$= T(xy)$

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with an averaging operator T
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$$\mu^1(x, y) = xT(y), \quad \mu^2(x, y) = T(x)y$$

$$\mu^{1,2}(x, y) = xy$$

is a di-$\mathcal{P}$-algebra. → tri-algebra

Corollary. For every di-$\mathcal{P}$-algebra D there exists a $\mathcal{P}$-algebra A with an averaging operator T such that D embeds into A. → tri-algebra

Replicating an operad
(PBW-type problems)

$$\omega : \mathcal{R} \to \mathcal{P} \text{ morphism of operads}$$

$$\text{functor} \quad \omega : \mathcal{P}\text{-algebras} \to \mathcal{R}\text{-algebras}$$
$$A \mapsto A^{(\omega)}$$

e.g.,

As $\to$ Lie
$A \mapsto A^{(-)}$, $[x, y] = xy - yx$

As $\to$ Jord
$A \mapsto A^{(+)}$, $x \cdot y = xy + yx$

? Whether every $\mathcal{R}$-algebra L may be embeded into $A^{(\omega)}$ for an appropriate $\mathcal{P}$-algebra A?
(Embedding problem)

? If $\dim L < \infty$, whether minimal $\dim A$ is finite?
(Ado problem)

? What is the structure of the universal enveloping $\mathcal{P}$-algebra $U(L)$ of a given $\mathcal{R}$-algebra L?
(Poincaré-Birkhoff-Witt problem)

Replicating an operad
(PBW-type problems)

A morphism $\omega : \mathcal{R} \to \mathcal{P}$

induces
$$\omega \otimes \mathrm{id} : \mathcal{R} \otimes \mathcal{C} \to \mathcal{P} \otimes \mathcal{C}$$
where $\mathcal{C} =$ Perm or ComTrias

Hence, we have natural functors

$\omega \otimes \mathrm{id}$: di-$\mathcal{P}$-algebras $\to$ di-$\mathcal{R}$-algebras

tri-$\mathcal{P}$-algebras $\to$ tri-$\mathcal{R}$-algebras

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tri-$\mathcal{P}$-algebras $\to$ tri-$\mathcal{R}$-algebras

Example:

$$\boxed{[x, y] = xy - yx}$$

$$\gamma(x, y) = \mu(x, y) - \mu^{(12)}(x, y) \mid \otimes e_1^{(2)} \in \mathrm{Perm}(2)$$

$$\gamma(x, y) \otimes e_1^{(2)} = \mu(x, y) \otimes e_1^{(2)} - \mu^{(12)}(x, y) \otimes e_1^{(2)}$$

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tri-$\mathcal{P}$-algebras $\to$ tri-$\mathcal{R}$-algebras

Example:

$$[x, y] = xy - yx$$

$$[x \dashv y] = x \dashv y - y \vdash x$$

$$\gamma(x, y) = \mu(x, y) - \mu^{(12)}(x, y) \mid \otimes e_1^{(2)} \in \mathrm{Perm}(2)$$

$$\gamma(x, y) \otimes e_1^{(2)} = \mu(x, y) \otimes e_1^{(2)} - \mu^{(12)}(x, y) \otimes e_1^{(2)}$$

$$\gamma(x, y) \otimes e_1^{(2)} = \gamma^1(x, y) = [x \dashv y]$$

$$\mu(x, y) \otimes e_1^{(2)} = \mu^1(x, y) = x \dashv y$$

$$\mu^{(12)}(x, y) \otimes e_1^{(2)} = (\mu(x, y) \otimes e_2^{(2)})^{(12)} = \mu^2(x, y)^{(12)} = y \vdash x$$

Replicating an operad
(PBW-type problems)

The same problems make sense in these settings

Replicating an operad
(PBW-type problems)

The same problems make sense in these settings

$$(C \otimes A)^{(\omega\otimes \mathrm{id})} \simeq C \otimes A^{(\omega)}$$

$C \in$ Perm or ComTrias

Corollary.

(1) If the Embedding problem has positive solution for $(\mathcal{R}, \mathcal{P}, \omega)$, then it has positive solution for (di-$\mathcal{R}$, di-$\mathcal{P}$, $\omega \otimes \mathrm{id}$) and (tri-$\mathcal{R}$, tri-$\mathcal{P}$, $\omega \otimes \mathrm{id}$);

(2) If the Ado problem has positive solution for $(\mathcal{R}, \mathcal{P}, \omega)$, then it has positive solution for (di-$\mathcal{R}$, di-$\mathcal{P}$, $\omega \otimes \mathrm{id}$) and (tri-$\mathcal{R}$, tri-$\mathcal{P}$, $\omega \otimes \mathrm{id}$);

(3) For a (di-)tri-$\mathcal{R}$-algebra L, the universal enveloping (di-)tri-$\mathcal{P}$-algebra $U(L)$ is isomorphic to the subalgebra of $C \otimes U(\widehat{L})$ generated by the image of L.

e.g., $C = \Bbbk\mathbb{Z}_2$

Replicating an operad
(Special identities)

$$\omega : \mathcal{R} \to \mathcal{P} \text{ morphism of operads}$$

$$\text{functor } \ \omega : \mathcal{P}\text{-algebras} \to \mathcal{R}\text{-algebras}$$

$$A \mapsto A^{(\omega)}$$

$\mathcal{R}$-algebra L is *special* if $L \subseteq A^{(\omega)}$

for some $\mathcal{P}$-algebra A.

$S\mathcal{R}$ = operad, governing the variety generated by all special $\mathcal{R}$-algebras

$$\mathcal{R} \to S\mathcal{R}$$

$$\text{Kernel} = \textit{Special identities}$$

Replicating an operad
(Special identities)

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$$\mathcal{R} \to S\mathcal{R}$$

$$\text{Kernel} = \textit{Special identities}$$

Example: $(\operatorname{char} \mathbb{k} \neq 2)$

$\mathcal{R} = \mathrm{Perm}$ $\qquad$ $\mathcal{P} = \mathrm{Com} + \text{Derivation } \partial\text{: } \partial^2(x) = 0$

$\omega : \mathcal{R} \to \mathcal{P}$ $\qquad$ $xy = \partial(x)y$

$S\mathcal{R}$-algebras satisfy $x(yz) = (xy)z = 0$

Replicating an operad
(Special identities)

Theorem (V. Gubarev, P.K., V. Voronin)

For any $\omega : \mathcal{R} \to \mathcal{P}$, consider

$$\omega \otimes \mathrm{id} : \text{di-}\mathcal{R} \to \text{di-}\mathcal{P}$$
$$\omega \otimes \mathrm{id} : \text{tri-}\mathcal{R} \to \text{tri-}\mathcal{P}$$

Then

$$S\text{di-}\mathcal{R} = \text{di-}S\mathcal{R}, \quad S\text{tri-}\mathcal{R} = \text{tri-}S\mathcal{R}.$$

Corollary.

(1) For di-As $\to$ di-Jord: No special identities of degree ≤ 7

[M. Bremner, L. Perezi, 2011]

(2) Special identities for di-As $\to$ di-JTS

[M. Bremner, R. Felipe, J. Sanchez-Ortega, 2011]

(3) Special identities for di-Lie = Leib $\to$ di-LTS

[M. Bremner, J. Sanchez-Ortega, 2011]

(4) For di-Alt $\to$ di-Mal: No special identities of degree < 7

[M. Bremner, L. Perezi, J. Sanchez-Ortega, 2011]

Splitting an operad

Algebra $(A, *)$

$(A, \prec, \succ)$: $x * y = x \prec y + x \succ y$ (pre-algebra)

$(A, \prec, \succ, \perp)$: $x * y = x \prec y + x \succ y + x \perp y$ (post-algebra)

in a coherent way

[C.Bai, O. Bellier, L. Guo, X. Ni, 2011] ⟶ name of the procedure

[V. Gubarev, P.K., 2011]

The structures obtained are ``dual'' to di- and tri-algebras

Splitting an operad

Koszul duality for operads [V. Ginzburg, M. Kapranov, 1994]

$\mathcal{P}$ binary quadratic operad

$f \in \Omega \Rightarrow \nu(f) = 2,$ $\mathcal{P}(2) \curvearrowleft S_2$ space of binary operations

$\dim \mathcal{P}(2) < \infty$

Defining identites have degree 3

$\mathcal{P}(3) = \Omega(3)/V$

$\Omega(3) =$ all polylinear terms of degree 3

$S_3 \curvearrowright V \subset \Omega(3)$

$\Omega^\vee(3) = \Omega(3)^* \curvearrowleft S_3$

skew-transpose action:

$\langle \varphi^\sigma, x \rangle = (-1)^\sigma \langle \varphi, x^\sigma \rangle$

$V^\vee \subset \Omega^\vee(3)$ dual subspace

$\mathcal{P}^!(3) = \Omega^\vee(3)/V^\vee$

$\mathcal{P}^!$ Koszul-dual operad

Example:

$\mathcal{P} = \mathrm{Lie}$

$\dim \mathcal{P}(2) = 1$ skew-symmetric

$\dim \mathcal{P}^\vee(2) = 1$ symmetric

$\dim \Omega(3) = 3$: $(xy)z,\ (yz)x,\ (zx)y$

Jacobi identity:

$$(xy)z + (yz)x + (zx)y = 0$$

dual space: $(xy)z = (yz)x = (zx)y$

$\mathcal{P}^! = \mathrm{Com}$

Splitting an operad

for binary quadratic operads

$\mathrm{Perm}^! = \text{pre-Lie}$

$\mathrm{ComTrias}^! = \text{post-Lie}$ [B. Vallette, 2007]

$\mathcal{P}$ binary quadratic operad

$$(\text{di-}\mathcal{P})^! = (\mathcal{P} \circ \mathrm{Perm})^! = \mathcal{P}^! \bullet \mathrm{Perm}^! = \mathcal{P}^! \bullet \text{pre-Lie}$$

$$(\text{tri-}\mathcal{P})^! = (\mathcal{P} \circ \mathrm{ComTrias})^! = \mathcal{P}^! \bullet \mathrm{ComTrias}^! = \mathcal{P}^! \bullet \text{post-Lie}$$

$\left.\begin{array}{l}\bullet\\ \circ\end{array}\right\}$ Black and white Manin products of binary quadratic operads

Splitting an operad
for binary quadratic operads

Com $\bullet$ pre-Lie = Zinb (Zinbiel algebras)
[J.-L. Loday, 1995]

As $\bullet$ pre-Lie = Dend (Dendriform algebras)
[J.-L. Loday, 2001]

Pois $\bullet$ pre-Lie = prePois (pre-Poisson algebras)
[M. Aguiar, 2000]

Dend $\bullet$ pre-Lie = Quad (Quadri-algebras)
[M. Aguiar, J.-L. Loday, 2004]

pre-Lie $\bullet$ pre-Lie = L-Dend (L-Dendriform algebras)
[C. Bai, L. Liu, X. Ni, 2010]

As $\bullet$ post-Lie = Tridend (Tridendriform algebras)
[J.-L. Loday, M. Ronco, 2004]

Com $\bullet$ post-Lie = CTD (Commutative tridendriform algebras)
[J.-L. Loday, 2007]

Splitting an operad
(Pre-algebras, general case)

$\Omega = \{f_1, f_2, \dots\}$
$\nu(f_k) = n_k \geq 1$

$\mathcal{P}$-Algebra:
(A, Ω^A)
$f_k^A : \underbrace{A \otimes \cdots \otimes A}_{n_k} \to A$

Defining identities Σ (multi-linear)

$\Longrightarrow$

$\Omega_2 = \{f_1^i, f_2^i, \dots\}$

$\nu(f_k^i) = n_k \geq 1$
$1 \leq i \leq n_k$

pre-$\mathcal{P}$-algebra
(D, Ω_2^D)

Defining identities pre-Σ_2:

$$P = \mathrm{Perm}\langle x_1, x_2, \dots \rangle$$

$P \otimes D$ as an Ω-algebra:

$$f(p_1 \otimes a_1, \dots, p_n \otimes a_n) = \sum_{i=1}^{n} e_i^{(n)}(p_1, \dots, p_n) \otimes f^i(a_1, \dots, a_n)$$

Definition:
D is a pre-$\mathcal{P}$-algebra iff $P \otimes D$ is a $\mathcal{P}$-algebra.

$$D \models \text{pre-}\Sigma_2 \iff P \otimes D \models \Sigma$$

Splitting an operad

Example: splitting of

$$\Sigma: \quad \boxed{[x, y, [u, v, w]] - [u, [x, y, v], w] = 0}$$

$$[\cdot, \cdot, \cdot] \Rightarrow [\cdot, \cdot, \cdot]_1, \, [\cdot, \cdot, \cdot]_2, \, [\cdot, \cdot, \cdot]_3$$

$$(D, \, [\cdot, \cdot, \cdot]_1, \, [\cdot, \cdot, \cdot]_2, \, [\cdot, \cdot, \cdot]_3)$$

Splitting an operad

Example: splitting of

$$\Sigma: \quad \boxed{[x,y,[u,v,w]] - [u,[x,y,v],w] = 0} \qquad \begin{cases} x = p \otimes a \\ y = q \otimes b \\ u = t \otimes c \\ v = r \otimes d \\ w = s \otimes e \end{cases}$$

$$[\cdot,\cdot,\cdot] \Rightarrow [\cdot,\cdot,\cdot]_1,\ [\cdot,\cdot,\cdot]_2,\ [\cdot,\cdot,\cdot]_3$$

$$p, q, r, s, t \in P \in \mathrm{Perm}$$

$$a, b, c, d, e \in D$$

$$(D,\ [\cdot,\cdot,\cdot]_1,\ [\cdot,\cdot,\cdot]_2,\ [\cdot,\cdot,\cdot]_3)$$

$$\boxed{[p \otimes a, q \otimes b, t \otimes c] = qtp \otimes [a,b,c]_1 + ptq \otimes [a,b,c]_2 + pqt \otimes [a,b,c]_3}$$

$$P \otimes D \models \Sigma \Rightarrow 5 \text{ identities } \text{pre-}\Sigma_2$$

Splitting an operad

(Post-algebras, general case)

$\Omega = \{f_1, f_2, \dots\}$

$\nu(f_k) = n_k \geq 1$

$\mathcal{P}$-Algebra:

(A, Ω^A)

$f_k^A : \underbrace{A \otimes \cdots \otimes A}_{n_k} \to A$

Defining identities Σ (multi-linear)

$\Longrightarrow$

$\Omega_3 = \{f_1^H, f_2^H, \dots\}$

$\nu(f_k^H) = n_k \geq 1$

$H \subseteq \{1, \dots, n_k\}$, $H \neq \varnothing$

post-$\mathcal{P}$-algebra

(D, Ω_3^D)

Defining identities post-Σ_3

$$C = \mathrm{ComTrias}\langle x_1, x_2, \dots\rangle$$

$C \otimes D$ as an Ω-algebra:

$$f(c_1 \otimes a_1, \dots, c_n \otimes a_n) = \sum_H e_H^{(n)}(c_1, \dots, c_n) \otimes f^H(a_1, \dots, a_n)$$

> Definition:
> D is a post-$\mathcal{P}$-algebra iff $C \otimes D$ is a $\mathcal{P}$-algebra.

$$D \models \text{pre-}\Sigma_3 \iff C \otimes D \models \Sigma$$

Splitting an operad

Theorem [V. Gubarev, P.K., 2011]
If $\mathcal{P}$ is a binary quadratic operad then

$$(\text{pre-}\mathcal{P})^! = \text{di-}\mathcal{P}^! \qquad (\text{post-}\mathcal{P})^! = \text{tri-}\mathcal{P}^!$$

Corollary

$$\text{pre-}\mathcal{P} = \mathcal{P} \bullet \text{pre-Lie}$$

$$\text{post-}\mathcal{P} = \mathcal{P} \bullet \text{post-Lie}$$

Splitting an operad

Corollary

If D is a (pre-)post-$\mathcal{P}$-algebra then $(C = \mathbb{k})$

for every (Perm-) ComTrias-algebra C

$C \otimes D$ is a $\mathcal{P}$-algebra.

Example:

(1) $\mathcal{P} = \mathrm{As}$, $\mu(x, y) = xy$

post-As = Tridend

$\mu^1(x, y) = x \prec y$, $\mu^2(x, y) = x \succ y$, $\mu^{1,2}(x, y) = x \perp y$

Then

$x * y = x \prec y + x \succ y + x \perp y$ is an associative product

(2) $\mathcal{P} = \mathrm{Lie}$, $\mu(x, y) = [x, y]$

pre-Lie $\mu^1(x, y) = [x \prec y]$, $\mu^2(x, y) = [x \succ y]$

$[x \prec y] = -[y \succ x]$, $xy := [x \succ y]$ left-symmetric product

Then

$x * y = [x \succ y] + [x \prec y] = xy - yx$ is a Lie product

Splitting an operad
(Rota-Baxter operators)

Demo case:
one binary operation
$\mu(x, y) = xy$

Definition:
$R : A \to A$ *Rota-Baxter operator* of weight 0
if $R(x)R(y) = R(R(x)y + xR(y))$
for $x, y \in A$

[G. Baxter, 1960]
[G.-C. Rota, 1969]

Theorem (c.f. [M. Aguiar, 2000])
If A is a $\mathcal{P}$-algebra with a RB-operator R
then A w.r.t.

$$\mu^1(x, y) = xR(y), \quad \mu^2(x, y) = R(x)y$$

is a pre-$\mathcal{P}$-algebra.

Splitting an operad
(Rota-Baxter operators)

Demo case:
one binary operation
$\mu(x, y) = xy$

Definition:
$R : A \to A$ *Rota-Baxter operator* of weight 1
if $R(x)R(y) = R(R(x)y + xR(y) + xy)$
for $x, y \in A$

Theorem (c.f. [K. Ebrahimi-Fard, 2002])
If A is a $\mathcal{P}$-algebra with a RB-operator R of weight 1
then A w.r.t.

$$\mu^1(x, y) = xR(y), \quad \mu^2(x, y) = R(x)y, \quad \mu^{1,2}(x, y) = xy$$

is a post-$\mathcal{P}$-algebra.

Splitting an operad
(Rota-Baxter operators)

$\mathcal{P}$ arbitrary operad

Theorem ([V. Gubarev, P.K., 2013])
If D is a (pre-) post-$\mathcal{P}$-algebra
then there exists a $\mathcal{P}$-algebra $\widehat{D}$
equipped with a Rota-Baxter operator R of weight (0) 1
such that D embeds into $\widehat{D}$

Splitting an operad
(Rota-Baxter operators)

$\mathcal{P}$ arbitrary operad

Theorem ([V. Gubarev, P.K., 2013])
If D is a (pre-) post-$\mathcal{P}$-algebra
then there exists a $\mathcal{P}$-algebra $\widehat{D}$
equipped with a Rota-Baxter operator R of weight (0) 1
such that D embeds into $\widehat{D}$

Proof: $\widehat{D} = \Bbbk\mathbb{Z}_2 \otimes D,\ \mathbb{Z}_2 = \{1, \tau\},\ \tau^2 = 1$

$$D \hookrightarrow \widehat{D}$$
$$a \mapsto (1 - \tau) \otimes a$$

Pre-:
$R(\tau \otimes a) = 1 \otimes a,\ R(1 \otimes a) = 0$: Rota-Baxter operator of weight 0

Post-:
$R(\tau \otimes a) = -2 \otimes a,\ R(1 \otimes a) = -1 \otimes a$: Rota-Baxter operator of weight 1

Corollary (c.f. [K. Ebrahimi-Fard, L. Guo, 2008])
The universal enveloping Rota-Baxter $\mathcal{P}$-algebra
of a free (pre-)post-$\mathcal{P}$-algebra is free.

Splitting an operad

(PBW-type problems and special identities)

Splitting an operad

(PBW-type problems and special identities)

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Splitting an operad
(PBW-type problems and special identities)

$$\omega : \mathcal{R} \to \mathcal{P} \text{ morphism of operads}$$

$f(x_1, \ldots, x_n) \in \mathcal{R}(n)$

$\omega(f)(x_1, \ldots, x_n) \in \mathcal{P}(n)$

$C = \mathrm{ComTrias}\langle c_1, c_2, \ldots \rangle$ (or Perm)

Then

$$\omega(f)(c_1 \otimes x_1, \ldots, c_n \otimes x_n) = \sum_{\substack{H \subseteq \{1,\ldots,n\} \\ H \neq \varnothing}} e_H^{(n)}(c_1, \ldots, c_n) \otimes g^H(x_1, \ldots, g_n)$$

$$\boxed{\text{post-}\omega(f^H) := g^H}$$ (or pre-ω)

Proposition.

pre-ω: pre-$\mathcal{R} \to$ pre-$\mathcal{P}$

post-ω: post-$\mathcal{R} \to$ post-$\mathcal{P}$

Splitting an operad

(Example)

$\mathcal{R} = LTS$ Lie triple systems

$\mathcal{P} =$ Lie

$\omega : [x, y, z] \mapsto [[x, y], z]$

pre-ω: converts a pre-Lie algebra into pre-LTS

$p, q, t \in P$ (Perm)

$$[p \otimes x, q \otimes y, t \otimes z] = pqt \otimes [x, y, z]_3 + ptq \otimes [x, y, z]_2 + qtp \otimes [x, y, z]_1$$

On the other hand,

$$[[p \otimes x, q \otimes y], t \otimes z] = [pq \otimes xy - qp \otimes yx, t \otimes z]$$

$$= pqt \otimes (xy)z - tpq \otimes z(xy) - qpt \otimes (yx)z + tqp \otimes z(yx)$$

$$= pqt \otimes [x, y]z + qtp \otimes z(yx) - ptq \otimes z(xy)$$

$(xy = [x \succ y] = -[y \prec x])$

Hence,

$$[x, y, z]_1 = z(yx)$$

$$[x, y, z]_2 = -z(xy) = -[y, x, z]_1$$

$$[x, y, z]_3 = (xy)z - (yx)z$$

Splitting an operad
(PBW-type problems and special identities)

$$\omega : \mathcal{R} \to \mathcal{P}$$
$$\text{pre-}\omega\text{: pre-}\mathcal{R} \to \text{pre-}\mathcal{P}$$
$$\text{post-}\omega\text{: post-}\mathcal{R} \to \text{post-}\mathcal{P}$$

Theorem.

Spre-$\mathcal{R}$-algebras $\subseteq$ pre-$S\mathcal{R}$-algebras

Spost-$\mathcal{R}$-algebras $\subseteq$ post-$S\mathcal{R}$-algebras

i.e., splitting of special identities

leads to special identites for pre- or post-algebras

[M. Bremner, S. Madariaga, 2013]: pre-Jord $\to$ pre-As
No special identities of degree <8
6 special identities of degree 8

Splitting an operad
(A counterexample)

Example: (char $\Bbbk \neq 2$)

$\mathcal{R} = \mathrm{Perm}$ $\qquad$ $\mathcal{P} = \mathrm{Com} + \text{Derivation } \partial\text{: } \partial^2(x) = 0$

$\omega : \mathcal{R} \to \mathcal{P}$ $\qquad$ $xy = \partial(x)y$

$$S\mathcal{R} = N3\text{:} \quad x(yz) = (xy)z = 0$$

Splitting:

pre-$N3$ is defined by

$$(x \prec y) \prec z = 0$$
$$(x \succ y) \prec z = 0$$
$$(x \prec y + x \succ y) \succ z = 0$$
$$x \prec (y \prec z + y \succ z) = 0$$
$$x \succ (y \prec z) = 0$$
$$x \succ (y \succ z) = 0$$

pre-$\mathcal{P}$ = Perm-algebra with a derivation ∂, $\partial^2 = 0$

pre-ω:

$x \succ y = \partial(x)y$, $x \prec y = y\partial(x)$

Then

$$x \succ y + y \prec x = \partial(x)y + x\partial(y) = \partial(xy)$$

so

$$\boxed{(x \succ y + y \prec x) \succ z = 0}$$

holds on all special pre-$\mathcal{R}$-algebras

Thank you!