Some gradings on nonassociative algebras related to fine gradings of exceptional simple Lie algebras

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Outline

Definition of a group grading

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- The *universal (abelian) group* $U(\Gamma)$ is the (abelian) group with generating set S and defining relations $s_1 s_2 = s_3$ whenever $0 \neq \mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$.

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We assume that $\dim \mathcal{A} < \infty$ and G is abelian.

Examples of gradings

Example

The following is a $\mathbb{Z}$-grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where

$$\mathfrak{g}_{-1} = \mathrm{Span}\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\},\ \mathfrak{g}_0 = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\},\ \mathfrak{g}_1 = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}.$$

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Example (Cartan grading)

Let $\mathfrak{g}$ be a s.s. Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\right)$$

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$\mathrm{Supp}\,\Gamma = \{0\} \cup \Phi$; $U(\Gamma) = G \cong \mathbb{Z}^r$ where $r = \dim \mathfrak{h}$.

Examples continued

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A grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

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Namely, $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ where $\mathbb{Z}_2^2 = \{e, a, b, c\}$ and

$$\mathfrak{g}_a = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}, \mathfrak{g}_b = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}, \mathfrak{g}_c = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}.$$

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Example (Generalized Pauli grading)

If $\varepsilon \in \mathbb{F}$, there is a grading on $\mathcal{R} = M_n(\mathbb{F})$ ($\Rightarrow$ on $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$) by $G = \mathbb{Z}_n^2$:

$$X = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \varepsilon^2 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \varepsilon^{n-1} \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where ε is a primitive n-th root of 1. Choose generators a and b of G and set $\mathcal{R}_{a^i b^j} = \mathbb{F} X^i Y^j$.

Isomorphism and equivalence of gradings

Definition

- Two G-gradings on $\mathcal{A}$, $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}'_g$, are *isomorphic* if there exists an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$.

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Example

All Pauli gradings on $M_n(\mathbb{F})$ or $\mathfrak{sl}_n(\mathbb{F})$ are equivalent. For $M_n(\mathbb{F})$, there are $\phi(n)$ (Euler function) non-isomorphic $\mathbb{Z}_n^2$-gradings among them. Hence $\frac{1}{2}\phi(n)$ for $\mathfrak{sl}_n(\mathbb{F})$ if $n > 2$.

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If $\mathbb{F}$ is a.c., $\operatorname{char}\mathbb{F} = 0$, then (equivalence classes of) fine gradings on $\mathcal{A}$ $\leftrightarrow$ (conjugacy classes of) maximal quasitori in $\operatorname{Aut}(\mathcal{A})$.

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For any $x, y \in \mathcal{A}$, define the operator $V_{x,y} \colon \mathcal{A} \to \mathcal{A}$ as follows:

$$V_{x,y}(z) = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y \quad \text{for all } z \in \mathcal{A},$$

and set $T_x = V_{x,1}$, i.e., $T_x(z) = xz + zx - z\bar{x}$.

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Definition (Allison, 1978)

A unital algebra with involution $(\mathcal{A}, \bar{\ })$ is said to be *structurable* if

$$[T_z, V_{x,y}] = V_{T_z(x),y} - V_{x,T_{\bar{z}}(y)} \quad \text{for all } x, y, z \in \mathcal{A}.$$

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If $(\mathcal{A}, ^{-})$ is structurable then it is *skew-alternative*, i.e.

$$(s, x, y) = -(x, s, y) = (x, y, s) \quad \text{for all } x, y, s \in \mathcal{A} \text{ with } \bar{s} = -s,$$

where $(x, y, z) := (xy)z - x(yz)$.

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Recall that a *Hurwitz algebra* is a unital algebra endowed with a nonsingular multiplicative quadratic form (the *norm*). The *standard conjugation* of a Hurwitz algebra $(\mathcal{C}, n)$ is given by $\bar{x} = -x + n(x, 1)1$.

Example

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are Hurwitz algebras then $(\mathcal{C}_1 \otimes \mathcal{C}_2, \bar{\ })$ is structurable where

$$\overline{x_1 \otimes x_2} = \bar{x}_1 \otimes \bar{x}_2 \quad \text{for all } x_1 \in \mathcal{C}_1 \text{ and } x_2 \in \mathcal{C}_2.$$

Lie algebras associated to a structurable algebra $\mathcal{A}$

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- The *structure Lie algebra* $\mathfrak{str}(\mathcal{A})$, which is the subalgebra of $\mathfrak{gl}(\mathcal{A})$ spanned by the operators $V_{x,y}$ for all $x, y \in \mathcal{A}$. For simple $\mathcal{A}$, $\mathrm{Der}(\mathcal{A})$ is a subalgebra of $\mathfrak{str}(\mathcal{A})$ and we have a $\mathbb{Z}_2$-grading on $\mathfrak{str}(\mathcal{A})$ with $\mathfrak{str}(\mathcal{A})_{\bar{0}} = \mathrm{Der}(\mathcal{A}) \oplus T_{\mathcal{S}}$ and $\mathfrak{str}(\mathcal{A})_{\bar{1}} = T_{\mathcal{H}}$.

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- The *Steinberg unitary Lie algebra* $\mathfrak{stu}_3(\mathcal{A})$ is obtained from three copies of $\mathcal{A}$.

Lie algebras associated to a structurable algebra $\mathcal{A}$

Let $\mathcal{H} = \{x \in \mathcal{A} \mid \bar{x} = x\}$ and $\mathcal{S} = \{x \in \mathcal{A} \mid \bar{x} = -x\}$.

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Any grading on $\mathcal{A}$ by an abelian group G induces a grading on

- $\mathrm{Der}(\mathcal{A})$ by G,
- $\mathfrak{str}(\mathcal{A})$ and its derived algebra $\mathfrak{str}_0(\mathcal{A})$ by $G \times \mathbb{Z}_2$,
- $\mathfrak{stu}_3(\mathcal{A})$ by $G \times \mathbb{Z}_2^2$,
- $Kan(\mathcal{A})$ by $G \times \mathbb{Z}$.

Cayley–Dickson doubling process

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 2$. Let $\mathcal{Q}$ be a Hurwitz algebra with norm n. Fix $0 \neq \alpha \in \mathbb{F}$ and let $\mathfrak{CD}(\mathcal{Q}, \alpha) = \mathcal{Q} \oplus \mathcal{Q}w$ be the direct sum of two copies of $\mathcal{Q}$, where we write the element (x, y) as $x + yw$, with multiplication

$$(a + bw)(c + dw) = (ac + \alpha\bar{d}b) + (da + b\bar{c})w,$$

and norm

$$n(x + yw) = n(x) - \alpha n(y).$$

It is well known that $\mathfrak{CD}(\mathcal{Q}, \alpha)$ is a Hurwitz algebra $\Leftrightarrow$ $\mathcal{Q}$ is associative.

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Note that $\mathcal{K} := \mathfrak{CD}(\mathbb{F}, \alpha)$ is $\mathbb{Z}_2$-graded, $\mathcal{Q} := \mathfrak{CD}(\mathcal{K}, \beta)$ is $\mathbb{Z}_2^2$-graded and $\mathcal{C} := \mathfrak{CD}(\mathcal{Q}, \gamma)$ is $\mathbb{Z}_2^3$-graded. Explicitly,

$$\mathcal{C} = \bigoplus_{\alpha \in \mathbb{Z}_2^3} \mathbb{F}e_\alpha \quad \text{where } e_\alpha = (w_1^{\alpha_1} w_2^{\alpha_2}) w_3^{\alpha_3}.$$

Division gradings and twisted group algebras

Thus, any Cayley algebra $\mathcal{C}$ can be realized as a twisted group algebra $\mathbb{F}^\sigma\mathbb{Z}_2^3$. If $\mathbb{F}$ is a.c. then w_i can be normalized (Albuquerque–Majid, 1999) so that $\sigma(\alpha,\beta)=(-1)^{\psi(\alpha,\beta)}$, where
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These gradings are *division gradings* in the sense that (nonzero) homogeneous elements are invertible.
If $\mathbb{F}$ is a.c. and $M_n(\mathbb{F})$ is endowed with a division grading by G then the support $T\subset G$ is a subgroup and $M_n(\mathbb{F})\cong\mathbb{F}^\sigma T$. Such gradings are classified up to isomorphism (Bahturin–K, 2010) by the pairs (T,β) where $\beta(a,b)=\sigma(a,b)/\sigma(b,a)$ is a nondegenerate alternating bicharacter $T\times T\to\mathbb{F}^\times$, $T\subset G$, $|T|=n^2$.

First Tits construction

Let $\mathbb{F}$ be an a.c. field, $\operatorname{char}\mathbb{F} \neq 2$. The simple exceptional Jordan algebra $\mathcal{A} = \mathcal{H}_3(\mathcal{C})$, with multiplication $x \circ y = \frac{1}{2}(xy + yx)$, can be realized as the sum of three copies of $\mathcal{R} = M_3(\mathbb{F})$.

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Any $x \in \mathcal{R}$ satisfies the Cayley–Hamilton equation

$$x^3 - \mathrm{tr}(x)x^2 + s(x)x - \det(x)1 = 0,$$

where $s(x) = \frac{1}{2}\big(\mathrm{tr}(x)^2 - \mathrm{tr}(x^2)\big)$. Define $x^\sharp = x^2 - \mathrm{tr}(x)x + s(x)1$, so $xx^\sharp = x^\sharp x = \det(x)1$ for any $x \in \mathcal{R}$, and its linearization

$$x \times y = \frac{1}{2}\Big(xy + yx - \big(\mathrm{tr}(x)y + \mathrm{tr}(y)x\big) + \big(\mathrm{tr}(x)\mathrm{tr}(y) - \mathrm{tr}(xy)\big)1\Big).$$

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Set $\bar{x} = x \times 1 = \frac{1}{2}\big(\mathrm{tr}(x)1 - x\big)$. Then $\mathcal{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2$, where $\mathcal{R}$ is linearly isomorphic to $\mathcal{R}_i$ ($x \mapsto x_i$), with the following multiplication:

	a'_0	b'_1	c'_2
a_0	$(a \circ a')_0$	$(\bar{a}b')_1$	$(c'\bar{a})_2$
b_1	$(\overline{a'}b)_1$	$(b \times b')_2$	$(\overline{bc'})_0$
c_2	$(c\overline{a'})_2$	$(\overline{b'c})_0$	$(c \times c')_1$

Albert algebra as a twisted group algebra

Assume $\operatorname{char}\mathbb{F} \neq 2, 3$ and let ω be a primitive cubic root of 1. Let

$$x = \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

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This is a division grading and identifies (Griess, 1990) $\mathcal{A}$ with $\mathbb{F}^\sigma \mathbb{Z}_3^3$ where

$$\sigma(\alpha, \beta) = \begin{cases} \omega^{\psi(\alpha,\beta)} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\psi(\alpha,\beta)} & \text{otherwise,} \end{cases}$$

and $\psi(\alpha, \beta) = (\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_3 - \beta_3)$.

Cayley–Dickson doubling process for Jordan algebras

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 2$. For a separable (finite-dimensional) Jordan algebra $(\mathcal{J}, \cdot)$ of degree 4 , we can define a structurable algebra by means of the following doubling process.

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Let $\mathcal{A} = \mathcal{J} \oplus v\mathcal{J}$ with multiplication determined by the following rules:

$$ab = a \cdot b,\ a(vb) = v(a^\theta \cdot b),\ (va)b = v(a^\theta \cdot b^\theta)^\theta,\ (va)(vb) = (a \cdot b^\theta)^\theta,$$

where $\theta \colon \mathcal{J} \to \mathcal{J}$ is a linear map defined by $1^\theta = 1$ and $a^\theta = -a$ for any element a whose generic trace is zero. The involution of $\mathcal{A}$ is defined by $\overline{a + vb} = a - vb^\theta$.

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Note that the only skew-symmetric elements are the scalar multiples of v. Since $v^2 = 1$, it follows that all automorphisms of $\mathcal{A}$ (commuting with involution) send v to $\pm v$ and all derivations (commuting with involution) send v to 0. Any automorphism or derivation of $\mathcal{J}$ extends uniquely to $\mathcal{A}$. A grading on $\mathcal{J}$ by an abelian group G induces a grading on $\mathcal{A}$ by $G \times \mathbb{Z}_2$.

The structurable algebra $\mathcal{H}_4(\mathcal{Q}) \oplus v\mathcal{H}_4(\mathcal{Q})$

Let $\mathcal{Q}$ be the split quaternion algebra over $\mathbb{F}$, equipped with its standard involution. Upon the identification $\mathcal{Q} \cong M_2(\mathbb{F})$, the involution switches E_{11} with E_{22} and multiplies both E_{12} and E_{21} by -1. The subalgebra $\mathcal{K} = \mathrm{Span}\,\{E_{11}, E_{22}\}$ is isomorphic to $\mathbb{F} \times \mathbb{F}$ with exchange involution.

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Consider the associative algebra $M_4(\mathcal{Q})$ with involution $(q_{ij})^* = (\overline{q}_{ji})$. Since $M_4(\mathcal{Q}) \cong M_4(\mathbb{F}) \otimes \mathcal{Q}$, we can alternatively write the elements of $M_4(\mathcal{Q})$ as sums of tensor products or as 2×2 matrices over $M_4(\mathbb{F})$.

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Consider the Jordan subalgebra of symmetric elements

$$\begin{aligned}\mathcal{H}_4(\mathcal{Q}) &= \{a \in M_4(\mathcal{Q}) \mid a^* = a\} \\ &= \{\left(\begin{smallmatrix} z & x \\ y & z^t \end{smallmatrix}\right) \mid x, y, z \in M_4(\mathbb{F}),\ x^t = -x,\ y^t = -y\}.\end{aligned}$$

Note that the subalgebra $\mathcal{H}_4(\mathcal{K}) \subset \mathcal{H}_4(\mathcal{Q})$ is isomorphic to $M_4(\mathbb{F})^{(+)}$.

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The Cayley–Dickson double $\mathcal{A} = \mathcal{H}_4(\mathcal{Q}) \oplus v\mathcal{H}_4(\mathcal{Q})$ is a simple structurable algebra of dimension 56. The simple Lie algebras of "series" E can be constructed in terms of $\mathcal{A}$ as follows: $\mathrm{Der}(\mathcal{A})$ has type E_6, $\mathfrak{str}_0(\mathcal{A})$ has type E_7 and $\mathfrak{stu}_3(\mathcal{A})$ has type E_8.

Construction of the $\mathbb{Z}_4^3$-grading

Assume $\mathbb{F}$ contains a 4-th root of 1. The construction will proceed in two steps:

- define a $\mathbb{Z}_4$-grading on $\mathcal{A} = \mathcal{H}_4(\mathcal{Q}) \oplus v\mathcal{H}_4(\mathcal{Q})$,
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$$\mathcal{A}_{\bar{1}} = \{x \otimes E_{12} + v(y \otimes E_{21}) \mid x, y \in M_4(\mathbb{F}),\ x^t = -x,\ y^t = -y\} \quad \text{and}$$
$$\mathcal{A}_{\bar{3}} = \{x \otimes E_{21} + v(y \otimes E_{12}) \mid x, y \in M_4(\mathbb{F}),\ x^t = -x,\ y^t = -y\} = v\mathcal{A}_{\bar{1}}.$$

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The group $\mathrm{GL}_4(\mathbb{F})$ acts on $\mathcal{H}_4(\mathcal{Q})$ via $g \mapsto \mathrm{Ad}\begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix}$. Let φ and ψ be the automorphisms of $\mathcal{H}_4(\mathcal{Q})$ corresponding to the generalized Pauli matrices X and Y in $\mathrm{GL}_4(\mathbb{F})$. We denote their extensions to $\mathcal{A}$ by the same symbols.

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Note that φ and ψ have order 4 and preserve the $\mathbb{Z}_4$-grading of $\mathcal{A}$

Construction of the $\mathbb{Z}_4^3$-grading

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- define a $\mathbb{Z}_4$-grading on $\mathcal{A} = \mathcal{H}_4(\mathcal{Q}) \oplus v\mathcal{H}_4(\mathcal{Q})$,
- refine it using two commuting automorphisms of order 4.

The even components of the $\mathbb{Z}_4$-grading are just $\mathcal{A}_{\bar{0}} = \mathcal{H}_4(\mathcal{K})$ and $\mathcal{A}_{\bar{2}} = v\mathcal{H}_4(\mathcal{K})$. The odd components are as follows:

$$\mathcal{A}_{\bar{1}} = \{x \otimes E_{12} + v(y \otimes E_{21}) \mid x, y \in M_4(\mathbb{F}),\ x^t = -x,\ y^t = -y\} \quad \text{and}$$
$$\mathcal{A}_{\bar{3}} = \{x \otimes E_{21} + v(y \otimes E_{12}) \mid x, y \in M_4(\mathbb{F}),\ x^t = -x,\ y^t = -y\} = v\mathcal{A}_{\bar{1}}.$$

The group $\mathrm{GL}_4(\mathbb{F})$ acts on $\mathcal{H}_4(\mathcal{Q})$ via $g \mapsto \mathrm{Ad}\begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix}$. Let φ and ψ be the automorphisms of $\mathcal{H}_4(\mathcal{Q})$ corresponding to the generalized Pauli matrices X and Y in $\mathrm{GL}_4(\mathbb{F})$. We denote their extensions to $\mathcal{A}$ by the same symbols.

Note that φ and ψ have order 4 and preserve the $\mathbb{Z}_4$-grading of $\mathcal{A}$, but they do not commute!

The automorphism π

The automorphisms φ and ψ of $\mathcal{A}$ commute on the even component $\mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{2}} = \mathcal{H}_4(\mathcal{K}) \oplus v\mathcal{H}_4(\mathcal{K})$ and anticommute on the odd component $\mathcal{A}_{\bar{1}} \oplus \mathcal{A}_{\bar{3}}$.

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We will construct another automorphism π of order 4 that preserves the $\mathbb{Z}_4$-grading, commutes with each of φ and ψ on $\mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{2}}$ and anticommutes on $\mathcal{A}_{\bar{1}} \oplus \mathcal{A}_{\bar{3}}$.

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Let $U = \{x \otimes E_{12} \mid x^t = -x\}$ and $V = \{y \otimes E_{21} \mid y^t = -y\}$, so $\mathcal{A}_{\bar{1}} = U \oplus vV$ and $\mathcal{A}_{\bar{3}} = V \oplus vU$.

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U and V are dual $\mathrm{GL}_4(\mathbb{F})$-modules, but isomorphic as $\mathrm{SL}_4(\mathbb{F})$-modules. We construct an $\mathrm{SL}_4(\mathbb{F})$-isomorphism $U \to V$, $x \otimes E_{12} \mapsto \widehat{x} \otimes E_{21}$, using the Pfaffian $\mathrm{pf}(x) = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$ for skew $x = (x_{ij}) \in M_4(\mathbb{F})$.

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Finally, we define $\pi \colon \mathcal{A} \to \mathcal{A}$ as identity on $\mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{2}}$ and

$$\begin{aligned} \pi(x \otimes E_{12}) &= -v(\widehat{x} \otimes E_{21}), & \pi(v(x \otimes E_{12})) &= -\widehat{x} \otimes E_{21}, \\ \pi(x \otimes E_{21}) &= v(\widehat{x} \otimes E_{12}), & \pi(v(x \otimes E_{21})) &= \widehat{x} \otimes E_{12}. \end{aligned}$$

Construction of the $\mathbb{Z}_4^3$-grading (continued)

Fix $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$, so $X = \mathrm{diag}(1, \mathbf{i}, -1, -\mathbf{i})$. We will keep ψ and replace φ by $\widetilde{\varphi}$, which is the composition of π and the action of $\widetilde{X} = \mathrm{diag}(\omega, \omega^3, \omega^5, \omega^7)$ where $\omega^2 = \mathbf{i}$. (We can temporarily extend $\mathbb{F}$.)

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Then $\widetilde{\varphi}$ and ψ are commuting automorphisms of order 4 and hence we get a $\mathbb{Z}_4^3$-grading of $\mathcal{A}$ by setting

$$\mathcal{A}_{(\bar{j},\bar{k},\bar{\ell})} = \{a \in \mathcal{A}_{\bar{j}} \mid \psi(a) = \mathbf{i}^k,\ \widetilde{\varphi}(a) = (-\mathbf{i})^\ell\}.$$

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Explicitly, the homogeneous components are given by

$$\mathcal{A}_{(\bar{0},\bar{k},\bar{\ell})} = \mathbb{F}(X^kY^\ell \otimes E_{11} + (X^kY^\ell)^t \otimes E_{22});$$
$$\mathcal{A}_{(\bar{2},\bar{k},\bar{\ell})} = \mathbb{F}v(X^kY^\ell \otimes E_{11} + (X^kY^\ell)^t \otimes E_{22});$$
$$\mathcal{A}_{(\bar{1},\bar{0},\bar{\ell})} = \mathbb{F}(\xi_1 \otimes E_{12} + \mathbf{i}^\ell v(\xi_1 \otimes E_{21})),\ \ell = 1, 3;$$
$$\mathcal{A}_{(\bar{1},\bar{2},\bar{\ell})} = \mathbb{F}(\xi_2 \otimes E_{12} + \mathbf{i}^\ell v(\xi_2 \otimes E_{21})),\ \ell = 1, 3;$$
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$$\mathcal{A}_{(\bar{1},\bar{3},\bar{\ell})} = \mathbb{F}(\xi_4 \otimes E_{12} - \mathbf{i}^\ell v(\xi_4 \otimes E_{21})),\ \ell = 1, 3;$$
$$\mathcal{A}_{(\bar{1},\bar{1},\bar{\ell})} = \mathbb{F}(\xi_5 \otimes E_{12} + \mathbf{i}^\ell v(\xi_5 \otimes E_{21})),\ \ell = 0, 2;$$
$$\mathcal{A}_{(\bar{1},\bar{3},\bar{\ell})} = \mathbb{F}(\xi_6 \otimes E_{12} + \mathbf{i}^\ell v(\xi_6 \otimes E_{21})),\ \ell = 0, 2.$$

Construction of the $\mathbb{Z}_4^3$-grading (completed)

On the previous slide, $\{\xi_1, \ldots, \xi_6\}$ is the following basis of $\mathcal{K}_4(\mathbb{F})$:

$$\xi_{1,2} = \begin{bmatrix} 0 & 1 & 0 & \mp 1 \\ & 0 & \pm 1 & 0 \\ & & 0 & 1 \\ \text{skew} & & & 0 \end{bmatrix}, \quad \xi_{3,4} = \begin{bmatrix} 0 & 1 & 0 & \pm \mathbf{i} \\ & 0 & \pm \mathbf{i} & 0 \\ & & 0 & -1 \\ \text{skew} & & & 0 \end{bmatrix}, \quad \xi_{5,6} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & \pm \mathbf{i} \\ & & 0 & 0 \\ \text{skew} & & & 0 \end{bmatrix}$$

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Recall that the even component $\mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{2}}$ is the double of the Jordan algebra $M_4(\mathbb{F})^{(+)}$, so this double receives a fine grading by $\mathbb{Z}_2 \times \mathbb{Z}_4^2$. Here the support is the entire group; there is a distinguished element $h = (\bar{1}, \bar{0}, \bar{0})$ of order 2 (the degree of v).

Fine gradings for G_2 and F_4

Assume that the ground field $\mathbb{F}$ is a.c., $\operatorname{char}\mathbb{F} \neq 2, 3$.

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Up to equivalence, there are exactly two fine gradings on the Cayley algebra $\mathcal{C}$: the division $\mathbb{Z}_2^3$-grading and the Cartan $\mathbb{Z}^2$-grading (Elduque, 1998). They yield two fine gradings on the simple Lie algebra $\mathrm{Der}(\mathcal{C})$ of type G_2, which is a complete list (Draper–Martin, 2006; Elduque–K, 2012).

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Up to equivalence, there are exactly four fine (abelian) gradings on the Albert algebra $\mathcal{J}$, with universal groups $\mathbb{Z}^4$, $\mathbb{Z} \times \mathbb{Z}_2^3$, $\mathbb{Z}_2^5$ and $\mathbb{Z}_3^3$ (Draper–Martin, 2009; Elduque–K, 2012). They yield four fine gradings on the simple Lie algebra $\mathrm{Der}(\mathcal{J})$ of type F_4, which is a complete list.

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Fine gradings for "series" E, infinite universal group

The ground field $\mathbb{F}$ is assumed a.c., $\mathrm{char}\,\mathbb{F} \neq 2, 3$.

E_6		E_7		E_8	
Universal group	Model	Universal group	Model	Universal group	Model
$\mathbb{Z}^6$	Cartan	$\mathbb{Z}^7$	Cartan	$\mathbb{Z}^8$	Cartan
$\mathbb{Z}^4 \times \mathbb{Z}_2$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{K})$	$\mathbb{Z}^4 \times \mathbb{Z}_2^2$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{Q})$	$\mathbb{Z}^4 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{\mathcal{A}})$ $(F_4\,,\ \mathcal{C})$
$\mathbb{Z}^2 \times \mathbb{Z}_3^2$	$\mathcal{T}(\Gamma^1_{\mathcal{C}}, \Gamma^2_{M_3(\mathbb{F})})$ $(G_2\,,\ M_3(\mathbb{F})^{(+)})$	—		$\mathbb{Z}^2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^1_{\mathcal{C}}, \Gamma^4_{\mathcal{A}})$ $(G_2\,,\ \mathcal{A})$
$\mathbb{Z}^2 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{M_3(\mathbb{F})})$ $(A_2\,,\ \mathcal{C})$	$\mathbb{Z}^3 \times \mathbb{Z}_2^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^1_{\mathcal{H}_3(\mathcal{Q})})$ $(C_3\,,\ \mathcal{C})$	—	
$\mathbb{Z}^2 \times \mathbb{Z}_2^3$	$Kan(\tilde{\Gamma}_{\mathcal{K}(\mathbb{F})})$ $(BC_2\,,\ \mathcal{K}(\mathbb{F})_{1/2})$	$\mathbb{Z}^2 \times \mathbb{Z}_2^4$	$Kan(\tilde{\Gamma}_{\mathcal{K}(\mathcal{K})})$ $(BC_2\,,\ \mathcal{K}(\mathcal{K})_{1/2})$	$\mathbb{Z}^2 \times \mathbb{Z}_2^5$	$Kan(\tilde{\Gamma}_{\mathcal{K}(\mathcal{Q})})$ $(BC_2\,,\ \mathcal{K}(\mathcal{Q})_{1/2})$
—		$\mathbb{Z} \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^1_{\mathcal{Q}}, \Gamma^4_{\mathcal{A}})$ $(A_1\,,\ \mathcal{A})$	—	
$\mathbb{Z} \times \mathbb{Z}_2^5$	$Kan(\Gamma^1_{\mathcal{K}(\mathbb{F})})$ $(BC_1\,,\ \mathcal{K}(\mathbb{F}))$	$\mathbb{Z} \times \mathbb{Z}_2^6$	$Kan(\Gamma^1_{\mathcal{K}(\mathcal{K})})$ $(BC_1\,,\ \mathcal{K}(\mathcal{K}))$	$\mathbb{Z} \times \mathbb{Z}_2^7$	$Kan(\Gamma^1_{\mathcal{K}(\mathcal{Q})})$ $(BC_1\,,\ \mathcal{K}(\mathcal{Q}))$
$\mathbb{Z} \times \mathbb{Z}_2^4$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{K} \otimes \mathcal{C})$	$\mathbb{Z} \times \mathbb{Z}_2^5$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{Q} \otimes \mathcal{C})$	$\mathbb{Z} \times \mathbb{Z}_2^6$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^3_{\mathcal{A}})$ $(BC_1\,,\ \mathcal{C} \otimes \mathcal{C})$
—		$\mathbb{Z} \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$Kan(\Gamma^2_{\mathcal{K}(\mathcal{K})})$ $(BC_1\,,\ \mathcal{K}(\mathcal{K}))$	$\mathbb{Z} \times \mathbb{Z}_4^3$	$Kan(\Gamma^2_{\mathcal{K}(\mathcal{Q})})$ $(BC_1\,,\ \mathcal{K}(\mathcal{Q}))$

Fine gradings for “series” E, finite universal group

E_6		E_7		E_8	
Universal group	Model	Universal group	Model	Universal group	Model
$\mathbb{Z}_3^4$	$\mathfrak{g}(\Gamma_{\bar{\mathcal{K}}}, \Gamma_{\mathcal{O}})$	——		$\mathbb{Z}_3^5$	$\mathfrak{g}(\Gamma_{\mathcal{O}}, \Gamma_{\mathcal{O}})$
$\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^2_{M_3(\mathbb{F})})$	——		——	
$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^4_{\mathcal{A}})$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^4_{\mathcal{A}})$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^4_{\mathcal{A}})$
$\mathbb{Z}_2^7$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathbb{F})})$	$\mathbb{Z}_2^8$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathcal{K})})$	$\mathbb{Z}_2^9$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathcal{Q})})$
$\mathbb{Z}_2^6$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^2_{\mathcal{A}})$	$\mathbb{Z}_2^7$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^2_{\mathcal{A}})$	$\mathbb{Z}_2^8$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^2_{\mathcal{A}})$
$\mathbb{Z}_4^3$	$\mathrm{Der}(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4^3 \times \mathbb{Z}_2$	$\mathfrak{str}_0(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4^3 \times \mathbb{Z}_2^2$	$\mathfrak{stu}_3(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$
$\mathbb{Z}_4 \times \mathbb{Z}_2^4$	$\mathrm{Der}(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4 \times \mathbb{Z}_2^5$	$\mathfrak{str}_0(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4 \times \mathbb{Z}_2^6$	$\mathfrak{stu}_3(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$
——		$\mathbb{Z}_4^2 \times \mathbb{Z}_2^3$	$\mathfrak{stu}_3(\Gamma^2_{\mathcal{X}(\mathcal{K})})$	——	
——		——		$\mathbb{Z}_5^3$	Jordan grading

Fine gradings for "series" E, finite universal group

E_6		E_7		E_8	
Universal group	Model	Universal group	Model	Universal group	Model
$\mathbb{Z}_3^4$	$\mathfrak{g}(\Gamma_{\bar{\mathcal{K}}}, \Gamma_{\mathcal{O}})$	——		$\mathbb{Z}_3^5$	$\mathfrak{g}(\Gamma_{\mathcal{O}}, \Gamma_{\mathcal{O}})$
$\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^2_{M_3(\mathbb{F})})$	——		——	
$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^4_{\mathcal{A}})$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^4_{\mathcal{A}})$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^3$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^4_{\mathcal{A}})$
$\mathbb{Z}_2^7$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathbb{F})})$	$\mathbb{Z}_2^8$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathcal{K})})$	$\mathbb{Z}_2^9$	$\mathfrak{stu}_3(\Gamma^1_{\mathcal{X}(\mathcal{Q})})$
$\mathbb{Z}_2^6$	$\mathcal{T}(\Gamma_{\mathcal{K}}, \Gamma^2_{\mathcal{A}})$	$\mathbb{Z}_2^7$	$\mathcal{T}(\Gamma^2_{\mathcal{Q}}, \Gamma^2_{\mathcal{A}})$	$\mathbb{Z}_2^8$	$\mathcal{T}(\Gamma^2_{\mathcal{C}}, \Gamma^2_{\mathcal{A}})$
$\mathbb{Z}_4^3$	$\mathrm{Der}(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4^3 \times \mathbb{Z}_2$	$\mathfrak{str}_0(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4^3 \times \mathbb{Z}_2^2$	$\mathfrak{stu}_3(\Gamma^2_{\mathcal{X}(\mathcal{Q})})$
$\mathbb{Z}_4 \times \mathbb{Z}_2^4$	$\mathrm{Der}(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4 \times \mathbb{Z}_2^5$	$\mathfrak{str}_0(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$	$\mathbb{Z}_4 \times \mathbb{Z}_2^6$	$\mathfrak{stu}_3(\Gamma^3_{\mathcal{X}(\mathcal{Q})})$
——		$\mathbb{Z}_4^2 \times \mathbb{Z}_2^3$	$\mathfrak{stu}_3(\Gamma^2_{\mathcal{X}(\mathcal{K})})$	——	
——		——		$\mathbb{Z}_5^3$	Jordan grading

The list is known to be complete (up to equivalence) for E_6 if $\mathrm{char}\,\mathbb{F} = 0$ (Draper–Viruel, preprint 2012).