Octonionic Ovoids

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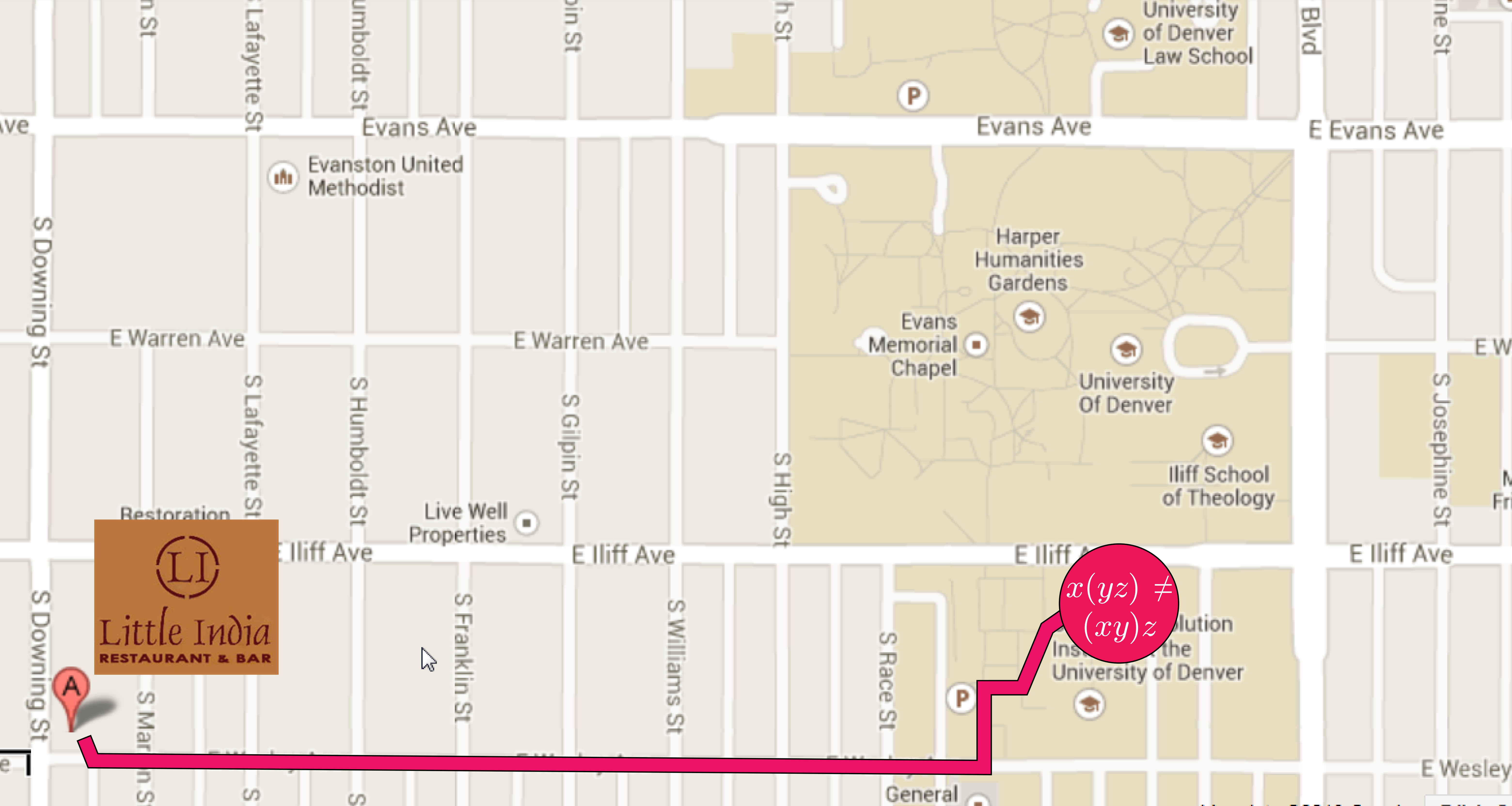

Evans Ave
E Evans Ave
University of Denver Law School
Evanston United Methodist
Harper Humanities Gardens
Evans Memorial Chapel
University Of Denver
Iliff School of Theology
E Warren Ave
S Downing St
S Lafayette St
S Humboldt St
S Gilpin St
S High St
S Josephine St
Live Well Properties
E Iliff Ave
S Franklin St
S Williams St
S Race St
University of Denver
E Wesley
General
Little India
RESTAURANT & BAR
$x(yz) \neq (xy)z$

Some ovoids in the $O_6^+(p)$ quadric (Klein quadric)

Consider a prime $p \equiv 1 \bmod 4$. Let $\mathcal{S}$ be the set of all $x = (x_1, \ldots, x_6) \in \mathbb{Z}^6$ such that

1. $x_i \equiv 1 \bmod 4$; and
2. $\sum_i x_i^2 = 6p$.

Then $|\mathcal{S}| = p^2 + 1$; and for all $x \neq y$ in $\mathcal{S}$, $x \cdot y \not\equiv 0 \bmod p$.

Example ($p = 5$, $|\mathcal{S}| = 5^2 + 1 = 26$)

$\mathcal{S}$ contains 6 vectors of shape $(5, 1, 1, 1, 1, 1)$;
20 vectors of shape $(-3, -3, -3, 1, 1, 1)$.

Example ($p = 13$, $|\mathcal{S}| = 13^2 + 1 = 170$)

$\mathcal{S}$ contains 20 vectors of shape $(5, 5, 5, 1, 1, 1)$;
30 vectors of shape $(-7, -5, 1, 1, 1, 1)$;
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Ovoids in $O^+_{2n}(q)$ quadrics

Let V be a $2n$-dimensional vector space over $\mathbb{F}_q$ with nondegenerate quadratic form $Q : V \to \mathbb{F}_q$.

(Projective) points are 1-dimensional subspaces $\langle v\rangle < V$; such a point is *singular* if $Q(v) = 0$. The associated *quadric* is the set of all singular points. A subspace $U \leqslant V$ is *totally singular* it lies entirely in the quadric, i.e. each of its points is singular. A *generator* is a maximal totally singular subspace. All generators have dimension n, if Q is chosen appropriately.

An *ovoid* is a set $\mathcal{O}$ of points of the quadric, meeting each generator exactly once. Equivalently, $\mathcal{O}$ is a set of $q^{n-1} + 1$ singular points, no two perpendicular.

The $O^+_4(q)$ quadric is a $(q + 1) \times (q + 1)$ grid; ovoids are transversals of the grid. Ovoids in the $O^+_6(q)$ quadric exist for all q. The lattice construction of ovoids in $O^+_6(p)$ (above) can be generalized to all primes p.

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Ovoids in $O_8^+(q)$ are known for *some* values of q, including all $q = p$ prime (Conway et al., 1988). No ovoids in $O_{2n}^+(q)$ are known in dimension $2n \geqslant 10$.

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The Ring O of Integral Octaves

Denote by O the *ring of integral octaves*. The octonion algebra is $\mathbb{O} = \mathbb{R} \otimes_{\mathbb{Z}} O$ and O is isometric to a root lattice of type E_8 in $\mathbb{O}$.

The set of units $\mathbb{O}^{\times}$ is a Moufang loop of order 240, consisting of all elements of norm 1 in O.

For all $n \geqslant 1$, the number of elements $v \in O$ of *norm* $|v|^2 = n$ is

$$240\sigma_3(n) = 240 \sum_{1 \leqslant d | n} d^3 .$$

Reduction mod p gives maps $\mathbb{Z} \to \mathbb{F}_p$ and $O \to V := O/pO$ denoted by $\overline{}$. Equipped with the quadratic form

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The ‘binary’ ovoids

Theorem (Conway, Kleidman & Wilson, 1988)

Let p be an odd prime. Fix a unit $u \in O^\times$. Let $\mathcal{S}$ be the set of vectors $x \in \mathbb{Z}u + 2O \subset O$ such that $|x|^2 = p$. Then $|\mathcal{S}| = 2(p^3+1)$ and $\mathcal{S}$ consists of $p^3 + 1$ pairs $\pm x$. Reducing these vectors mod pO gives

$$\mathcal{O} = \mathcal{O}_{2,p,u} = \{\langle \overline{x} \rangle \ : \ \pm x \in \mathcal{S}\},$$

an ovoid in $O/pO \simeq O_8^+(p)$.

The proof uses the most basic facts about the E_8 root lattice. Conway et al. also gave a construction of ‘ternary’ ovoids (replacing the prime 2 by 3 above).

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The r-ary ovoids in $O_8^+(p)$

Theorem (M., 1993)

Let $r \neq p$ be odd primes. Fix $u \in O$ such that $\left(\frac{-p|u|^2}{r}\right) = +1$. Let $\mathcal{S}$ be the set of vectors $x \in \mathbb{Z}u + rO \subset O$ such that $|x|^2 = k(r-k)p$ for some $k \in \{1, 2, \ldots, \frac{r-1}{2}\}$. Then $|\mathcal{S}| = 2(p^3+1)$ and $\mathcal{S}$ consists of p^3+1 pairs $\pm x$. Reducing these vectors mod pO gives

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The proof uses facts about E_8 *and* the fact that $E_8 \oplus E_8$ has $480\sigma_7(n)$ elements of norm $n \geqslant 1$. (*Or* O and theorems on factorization in O). Ovoids isomorphic to $\mathcal{O}_{r,p,u}$ (for primes $r \neq p$, including $r = 2$) are the *r-ary ovoids of octonionic type* in $O_8^+(p)$.

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Open Questions

1. For each p, there are infinitely many choices of r, u to choose in constructing $\mathcal{O}_{r,p,u}$ but only finitely many ovoids in $O_8^+(p)$. How many? How do we know when we have found them all?
2. Let $w(p)$ be the number of isomorphism classes of *octonionic ovoids* in $O_8^+(p)$. Does $w(p) \to \infty$ as $p \to \infty$? (By Conway et al. (1988), $w(p) \geqslant 1$.)
3. r, p don't really have to be primes. Does anything comparable work in $O_8^+(q)$?
4. Most octonionic ovoids should be rigid, i.e. having trivial stabilizer in $PGO_8^+(p)$; but no rigid ovoids in $O_8^+(q)$ have been found.
5. What is *really going on* in the construction of octonionic ovoids?

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Conjectured number of octonionic ovoids

Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_w$ be representatives for the isomorphism types of octonionic ovoids in $O_8^+(p)$, under $G = PGO_8^+(p)$. The number of ovoids isomorphic to $\mathcal{O}_i$ is $[G : G_{\mathcal{O}_i}]$; note that

$$|G| = |PGO_8^+(p)| = \tfrac{2}{d}p^{12}(p^6-1)(p^4-1)^2(p^2-1)$$

where $d = \gcd(p-1, 2)$.

The subgroup $W(E_8)/\{\pm I\} \cong PGO_8^+(2) \leqslant G$ has order

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Conjectured Mass Formula

For $p \geqslant 5$,

$$\sum_{i=1}^{w(p)} [G : G_{\mathcal{O}_i}] = \frac{|G|(p^4+239)}{4|PGO_8^+(2)|};$$

i.e.

$$\frac{|PGO_8^+(2)|}{|G_{\mathcal{O}_1}|} + \frac{|PGO_8^+(2)|}{|G_{\mathcal{O}_2}|} + \cdots + \frac{|PGO_8^+(2)|}{|G_{\mathcal{O}_w}|} = \frac{p^4+239}{4}.$$

The stabilizers $G_{\mathcal{O}_i}$ are not necessarily subgroups of $PGO_8^+(2)$. I am not claiming that the terms in this sum are always integers (but in every known case they are).

The cases $p = 2, 3$ are genuine exceptions. (When $p = 3$ the octonionic ovoids lie in hyperplanes.)

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The abundance of ovoids

Corollary

Let $n(p)$ be the number of isomorphism types of ovoids in $O_8^+(p)$. If the Mass Formula holds, then for some absolute constant $C > 0$, $n(p) \geqslant Cp^4 \to \infty$ as $p \to \infty$.

Currently it is known that $n(p) \geqslant 1$ (Conway et al., 1988).

Verifying the Mass Formula for small p

p	$w(p)$	Mass Formula
5	2	$96+120 = 216 = \frac{5^4+239}{4}$
7	2	$120+540 = 660 = \frac{7^4+239}{4}$
11	4	$120+120+960+2520 = 3720 = \frac{11^4+239}{4}$
13	4	$120+1080+1680+4320 = 7200 = \frac{13^4+239}{4}$
17	7	$120+120+540+960+3360+4320+11520 = 20940 = \frac{17^4+239}{4}$
19	6	$120+120+1080+7560+8640+15120 = 32640 = \frac{19^4+239}{4}$
23	10	$120+120+120+540+960+2520+3360$ $+7560+20160+34560 = 70020 = \frac{23^4+239}{4}$

Strictly speaking, these terms are *lower bounds* found by enumerating r-ary ovoids in $O_8^+(p)$ for small r and testing for isomorphism. To compute $\mathrm{Aut}(\mathcal{O})$, use `nauty` to determine $\mathrm{Aut}(\Delta(\mathcal{O}))$ where $\Delta(\mathcal{O})$ is the associated two-graph. In general $\mathrm{Aut}(\mathcal{O}) \subseteq \mathrm{Aut}(\Delta(\mathcal{O}))$, and we check that equality holds in all cases.

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Canonical bijections between octonionic ovoids in $O_8^+(p)$

Fix odd primes $r \neq p$ and $u \in O$ such that $\left(\frac{-p|u|^2}{r}\right) = +1$.

Denote the binary ovoid

$$\mathcal{O}_{2,p,1} = \left\{\langle \overline{x} \rangle \,:\, \pm x \in \mathbb{Z} + 2O,\ |x|^2 = p\right\}.$$

An alternative construction of the r-ary ovoid $\mathcal{O}_{r,p,u}$ is via the canonical bijection

$$f : \mathcal{O}_{r,p,u} \to \mathcal{O}_{2,p,1}$$

constructed as follows. Given $w \in \mathbb{Z}u + rO$ with $|x|^2 = k(r-k)p$, $1 \leqslant k \leqslant \frac{r-1}{2}$, we have

$$w = xy$$

for some $x, y \in O$ such that $|x|^2 = p$ and $|y|^2 = k(r-k)$. If we also require $x \in \mathbb{Z} + 2O$, then this factorization is unique up to a ± 1 factor and our bijection is

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Thank You!

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