The upper triangular algebra loop of degree 4

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Introduction

We define a natural loop structure on the set U_4 of unimodular upper-triangular matrices over a given field. It is shown that the loop is non-associative and nilpotent, of class 3. The conjugacy classes are computed and its seen that they lie between group conjugacy classes and superclasses.

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Quasigroups

Definition

A **quasigroup**, written as Q or $(Q, \cdot, /, \backslash)$ is a set Q equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{llll} \text{(SL)} & y \cdot (y \backslash x) = x\,; & \text{(SR)} & x = (x/y) \cdot y\ : \\ \text{(IL)} & y \backslash (y \cdot x) = x\,; & \text{(IR)} & x = (x \cdot y)/y\,. \end{array}$$

Loops

Definition

A **Loop**, is a quasigroup Q with an *identity* element 1 such that $1 \cdot x = x = x \cdot 1$ for all x in Q.

Loop conjugacy classes

The *inner multiplication group* $\mathrm{Inn}\, Q$ is the stabilizer $\mathrm{Mlt}\, Q_1$ in $\mathrm{Mlt}\, Q$ of the identity element 1.
For example, if Q is a group, then $\mathrm{Inn}\, Q$ is the inner automorphism group of Q, although for a general loop Q, elements of $\mathrm{Inn}\, Q$ are not necessarily automorphisms of Q.

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For elements q, r of Q, define the *conjugation*

$$T(q) = R(q)L(q)^{-1}\,, \tag{1}$$

the *right inner mapping*

$$R(q,r) = R(q)R(r)R(qr)^{-1}\,, \tag{2}$$

and the *left inner mapping*

$$L(q,r) = L(q)L(r)L(rq)^{-1} \tag{3}$$

in $\mathrm{Mlt}\, Q_1$.

Collectively, (1)-(3) are known as *inner mappings*.
The orbits of $\operatorname{Inn} Q$ on Q are defined as the (*loop*) *conjugacy classes* of Q.
If Q is a group, the loop conjugacy classes of Q are just the usual group conjugacy classes.

The loop multiplication

Consider matrices

$$x = \begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 1 & y_{12} & y_{13} & y_{14} \\ 0 & 1 & y_{23} & y_{24} \\ 0 & 0 & 1 & y_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with entries x_{ij}, y_{ij} from a field F.
We define the set of such matrices as U_4.

Then the quasigroup product is

$$\begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y_{12} & y_{13} & y_{14} \\ 0 & 1 & y_{23} & y_{24} \\ 0 & 0 & 1 & y_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 1 & x_{12}+y_{12} & x_{13}+y_{13}+x_{12}y_{23} & [x\cdot y]_{14} \\ 0 & 1 & x_{23}+y_{23} & x_{24}+y_{24}+x_{23}y_{34} \\ 0 & 0 & 1 & x_{34}+y_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with

$$[x\cdot y]_{14} = x_{14}+y_{14}+x_{12}y_{24}+x_{13}y_{34}+x_{12}x_{23}y_{34}+x_{12}y_{23}y_{34} \tag{5}$$

as the (1,4)-th entry of the product

The summands in (5) correspond to paths of respective lengths $1, 2, 3$ from 1 to 4 in the chain $1 < 2 < 3 < 4$, with labels chosen from x over the former part of the path, and y over the latter part. The other entries in the product have a similar (but simpler) structure.

Note that the above product has the matrix I_4 as its identity element.

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The right division

With matrices x and y as above, consider the right division $z = y/x$, namely a solution

$$z = \begin{bmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to the equation $z \cdot x = y$.

Lemma

There is a unique solution $z = yR(x)^{-1}$ to $z \cdot x = y$.

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Proof.

The entries z_{ij}, for $1 \leq i < j \leq 4$, are obtained by recursion on the length $j - i$ of the path from i to j in the chain $1 < 2 < 3 < 4$.

Paths of length 1:

$$z_{12} = (y_{12} - x_{12}) , \; z_{23} = (y_{23} - x_{23}) , \; z_{34} = (y_{34} - x_{34}) .$$

Paths of length 2:

$$z_{13} + x_{13} + z_{12}x_{23} = y_{13} , \text{ so}$$

$$z_{13} = y_{13} - x_{13} - z_{12}x_{23} = (y_{13} - x_{13}) - (y_{12} - x_{12})x_{23} .$$

Similarly, $z_{24} = (y_{24} - x_{24}) - (y_{23} - x_{23})x_{34} .$

□

Proof (Cont.)

The path of length 3:

$$z_{14} + x_{14} + z_{12}x_{24} + z_{13}x_{34} + z_{12}z_{23}x_{34} + z_{12}x_{23}x_{34} = y_{14}\,, \text{ so}$$

$$\begin{aligned} z_{14} &= y_{14} - x_{14} - z_{12}x_{24} - z_{13}x_{34} - z_{12}z_{23}x_{34} - z_{12}x_{23}x_{34} \\ &= (y_{14} - x_{14}) \\ &+ (y_{12} - x_{12})(-x_{24}) + (y_{13} - x_{13})(-x_{34}) \\ &+ (y_{12} - x_{12})(y_{23} - x_{23})(-x_{34})\,. \end{aligned}$$

Note that in each case, the coefficient z_{ij} is uniquely specified in terms of x and y. □

The left division

With matrices x and y as above, consider the left division $z = x\backslash y$, namely a solution

$$z = \begin{bmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Proof.

The proof is similar to the right division lemma. □

The algebra loop

Proposition

With the product (4), the algebra group U_4 over a field F forms a loop.

Definition

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Inner mappings

Within the loop U_4, the effects of the inner mappings (1)-(3) may be computed using the work of (4), the right division and the left division. Consider elements $x = [x_{ij}], q = [q_{ij}], r = [r_{ij}]$ of U_4. Then

$$xT(q) = xR(q)L(q)^{-1} = q\backslash xq$$

$$= \begin{bmatrix} 1 & x_{12} & x_{13} + \begin{vmatrix} x_{12} & q_{12} \\ x_{23} & q_{23} \end{vmatrix} & [xT(q)]_{14} \\ 0 & 1 & x_{23} & x_{24} + \begin{vmatrix} x_{23} & q_{23} \\ x_{34} & q_{34} \end{vmatrix} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $[xT(q)]_{14} =$

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Furthermore, one has

$$[xR(q,r)]_{14} = [(xq \cdot r)/qr]_{14} = x_{14} + q_{12}x_{23}r_{34} - x_{12}r_{23}q_{34}$$

and

$$[xL(q,r)]_{14} = [rq\backslash(r \cdot qx)]_{14} = x_{14} + r_{12}x_{23}q_{34} - q_{12}r_{23}x_{34} ,$$

with

$$[xR(q,r)]_{ij} = [xL(q,r)]_{ij} = x_{ij}$$

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for $1 \leq i < j \leq 4$ and $j - i < 3$.

Properties of the algebra loop

Definition

For given $1 \leq l < m \leq 4$, the *elementary* element E^{lm} of U_4 is defined by

$$[E^{lm}]_{ij} = \begin{cases} 1 & \text{if } i = l \text{ and } j = m, \text{ or if } i = j\,; \\ 0 & \text{otherwise.} \end{cases}$$

Properties of the algebra loop (Cont.)

Theorem

Over a given field F, the upper triangular algebra loop U_4 of degree 4 is neither commutative nor associative.

Proof.

It was already observed that U_4 forms a loop. Consider elementary elements $x = E^{23}$, $q = E^{12}$, $r = E^{34}$ of U_4. The computations of the inner mappings show that

$$[xT(q)]_{13} = x_{13} + \begin{vmatrix} x_{12} & q_{12} \\ x_{23} & q_{23} \end{vmatrix} = -1 \neq 0 = x_{13}\,,$$

so the loop is not commutative, and

$$[xR(q,r)]_{14} = x_{14} + q_{12}x_{23}r_{34} - x_{12}r_{23}q_{34} = 1 \neq 0 = x_{14}\,,$$

so the loop is not associative. □

Nilpotence and Conjugacy Classes

Consider the upper triangular algebra loop U_4 of degree 4 over a given field F. We demonstrate that U_4 is nilpotent, and determine the loop conjugacy class of each element $x = [x_{ij}]$ of U_4.

Recall that the *center* Z or $Z(Q)$ of a loop Q is the set

$$\{z \in Q \mid \forall\ q, r \in Q\ ,\ zT(q) = zR(q,r) = zL(q,r) = z\}$$

In other words, the center $Z(Q)$ consists precisely of the set of elements of Q which lie in singleton conjugacy classes.

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Center of U_4

Proposition

The set

$$\{x = [x_{ij}] \in U_4 \mid x_{ij} = 0 \text{ if } 1 \leq j - i < 3\} \tag{6}$$

forms the center of U_4.

Thus the center of U_4 is the set

$$Z(Q) = \left\{ \begin{bmatrix} 1 & 0 & 0 & x_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, x_{14} \in F \right\}$$

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Nilpotence of U_4

Theorem

The loop U_4 is nilpotent, of class 3. Indeed,

$$Z_{4-k}(U_4) = \{x = [x_{ij}] \in U_4 \mid x_{ij} = 0 \text{ if } 1 \leq j - i < k\} \tag{7}$$

for $1 \leq k \leq 4$.

Zeroes on the superdiagonal

Proposition

If the vector (x_{13}, x_{24}) is non-zero, the conjugacy class of

$$x = \begin{bmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is

$$\left\{ \begin{bmatrix} 1 & 0 & x_{13} & a \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \, a \in F \right\}. \tag{8}$$

The remaining cases

Proposition

If x_{23} is non-zero, the conjugacy class of x is

$$\left\{ \begin{bmatrix} 1 & x_{12} & b & a \\ 0 & 1 & x_{23} & c \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \, a, b, c \in F \right\} . \tag{9}$$

The remaining cases(Cont.)

Proposition

Suppose that $x_{23} = 0$, while the vector (x_{12}, x_{34}) is non-zero. Then the conjugacy class of x is

$$\left\{ \begin{bmatrix} 1 & x_{12} & x_{13} + bx_{12} & a \\ 0 & 1 & 0 & x_{24} - bx_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \, a, b \in F \right\} . \tag{10}$$

The remaining cases(Cont.)

Corollary

Suppose that $x_{23} = 0$, while the vector (x_{12}, x_{34}) is non-zero. Then the conjugacy class of x has cardinality $|F|^2$.

Remark

The Corollary shows that if F is finite, the size of the loop conjugacy class of $E^{12} + E^{34} - 1$ is $|F|^2$. On the other hand, the computations (in the language of Diaconis/Isaacs) show that the superclass of $E^{12} + E^{34} - 1$ has size $|F|^3$. Thus the loop conjugacy classes in U_4 do not necessarily coincide with the superclasses.

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Type of element	Size of class	Number of classes
$\begin{matrix}0 & 0 & * \\ & 0 & 0 \\ & & 0\end{matrix}$	1	q
$\begin{matrix}0 & * & F \\ & 0 & * \\ & & 0\end{matrix}$	q	q^2-1
$\begin{matrix}* & F & F \\ & * \neq 0 & F \\ & & *\end{matrix}$	q^3	$q^2(q-1)$
$\begin{matrix}* & F & F \\ & 0 & F\end{matrix}$	q^2	$q(q^2-1)$

Summary

The Table lists the sizes and number of each kind of loop conjugacy class in U_4. The element types are identified by the pattern of matrix entries above the diagonal, in conjunction with the reference to the proposition giving the full description of the type. The symbol $*$ is used as a "wild card" to denote a (potentially) non-zero field element. As a pattern entry, the symbol F denotes arbitrary elements of F that appear in the class. The symbol q stands for the cardinality of the underlying field F.

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