The Sabinin product in loops and quasigroups

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This talk is dedicated to the memory of

Lev Vasil’evich Sabinin

21 June 1932 – 4 June 2004
Summary

1. Left loops and their multiplication group
2. Loops and transitive permutation groups
3. The holomorph
3. Loops and transitive permutation groups
4. Sabinin’s product
5. Groups with a triality

These pages are just a guide for a spoken text. From a certain point on the presentation switches from left multiplications to right multiplications.
A magma\(^1\) \((M, \circ)\) is a set \(M\) with a multiplication \(\circ\) and a **left loop** is a universal algebra

\[ L = \langle L, \setminus, \circ, 1_r \rangle \]

of type \((2, 0, 1)\) satisfying the identities

\[
\begin{align*}
    a \circ (a \setminus b) &= b, \\
    a \setminus (a \circ b) &= b, \\
    a \circ 1_r &= a.
\end{align*}
\]

In a left loop \(Q\) the **left translations** defined by \(L_a(x) = ax\) are bijections. Call the group \(\text{LMult}(Q)\) generated by the set \(T_\ell(Q)\) of all left translations of \(Q\) the **left multiplication group** of \(Q\).

**Note:** \((L_a)^{-1}(x) = a \setminus x.\)

\(^1\)The name "groupoid" is taboo.
If $Q$ is a left loop, we have the transitive action of the group $\text{LMult}(Q)$ on the set $Q$. In Sabinin’s terminology the stabilizer $\mathcal{I}_\ell(Q) = \text{Stab}_{\text{LMult}(Q)}(1_r) = \{X \in \text{LMult}(Q) | X(1_r) = 1_r\}$ is called the left associant of $Q$. Others call $\mathcal{I}_\ell(Q)$ the left inner mapping group of $Q$.

Note that for any $a \in Q$ one has

$$\text{Stab}_{\text{LMult}(Q)}(a) = L_a\text{Stab}_{\text{LMult}(Q)}(1_r)L_a^{-1}. \quad (1)$$
Right loops

Given a right loop $L = \langle \circ, /, 1_{\ell} \rangle$ we have

- Right multiplications $R_b$
- The set of all right translations $T_r(Q)$
- The right multiplication group $RMult(Q) = \langle T_r(Q) \rangle$
- The right associant $I_r(Q) = Stab_{RMult(Q)}(1_{\ell})$
Loops

A **loop** is at the same time a left and a right loop. Note that left and right neutral element coincide. So in a loop $L = \langle \circ, \setminus, /, 1 \rangle$ we have

- $T_{\ell}(Q), L\text{Mult}(Q), I_{\ell}(Q)$
- $T_{r}(Q), R\text{Mult}(Q), I_{r}(Q)$
- The multiplication group $\text{Mult}(Q)$
- The associant $I(Q) = \text{Stab}_{R\text{Mult}(Q)}(1)$
For left, right or twosided quasigroup $Q$ in the definition of the associants one chooses an arbitrary base point $x_0 \in Q$ and observes equation 1.

I will not follow that path in my talk.
Given a transitive action $\eta: G \times X \to X$ of a group $G$ on a set $X$ put $H = H_{\eta,x_0} = \text{Stab}_G(x_0)$ for an arbitrary base point $x_0 \in X$.

Now identify $X$ with coset space $G/H$ via
\[ g_1x_0 = g_2x_0 \iff x_0 = g_1^{-1}g_2x_0 \iff g_1^{-1}g_2 \in H \iff g_1H = g_2H \]
and choose a coset representative system $K$ of $H$ in $G$.

We consider triples $\mathcal{B} = (G, H, K)$ - group, subgroup, transversal – and call them \textbf{Baer triples}.

Now one defines a multiplication $\diamond_B$ on the set $K$ by
\[(xH)(yH) = (x \diamond_B y)H.\]

\textbf{Theorem}

[BAER] \textit{For a Baer triple $\mathcal{B}$ the magma $(K, \diamond_B)$ is a left loop.}
Baers construction

We denote the left loop defined in the last theorem by $Q_B$. Note that we are not planning to discuss its dependence on the transversal $K$. However, there are some useful observations.

(B.1) $\bigcap_{g \in G} H^g = 1$ if and only if the action of $G$ is faithful.

(B.2) In $Q_B$ there is a twosided neutral element if $1_G \in K$.

(B.3) $Q_B$ is a loop if and only if $K$ is a transversal of $H^g$ for all $g \in G$.

(B.4) $G = \text{LMult}(Q_B)$ if and only if $K$ is a generating set of the group $G$.

Theorem

[BAER] For a left loop $Q$ and the Baer triple $\mathcal{B} = (\text{LMult}(Q), \mathcal{I}_\ell(Q), \mathcal{T}_\ell(Q))$ the left loop $Q_B$ isomorphic to $Q$. 
The torsor of a group


For group $G$ one calls the ternary operation

$$\tau_G(a, b, c) = ab^{-1}c$$

the torsor of $G$. One easily sees

**Proposition**

*If $X$ is a coset of some subgroup of a group $G$, then $\tau_G(X, X, X) \subseteq X$.***

An origin of the notion of the torsor lies in affine geometry.
The holomorph of a group

Let $G$ be a group. Then a bijection $\beta : G \to G$ is called a **holomorphism** if $\tau_G(a^\beta, b^\beta, c^\beta) = \tau_G(a, b, c)^\beta$ for all $a, b, c \in G$. The set $\text{Hol}(G)$ of all holomorphisms of $G$ forms a group, the **holomorph** of $G$.

**Theorem**

*For any group $G$ one has $\text{LMult}(G), \text{RMult}(G) \triangleleft \text{Hol}(G)$ and $\text{Aut}(G) \leq \text{Hol}(G)$. Furthermore, $\text{Inn}(G) \leq \text{Mult}(G)$ and

$$\text{Hol}(G) = \text{Aut}(G) \rtimes \text{LMult}(G) = \text{Aut}(G) \rtimes \text{RMult}(G) \cong \text{Aut}(G) \rtimes G.$$*
The holomorph of a loop

Theorem
For a loop \((Q, \cdot)\) and a group \(\Theta\) acting – not necessarily faithfully – as a group of permutations on the set \(Q\) by

\[
(A, x) \ast (B, y) = (AB, xB \cdot y)
\]  \hspace{1cm} (2)

a multiplication is defined on the set \(\Theta \times Q\). One has:

(i) \((\Theta \times Q, \ast)\) is a quasigroup with the left neutral element \((\text{id}_Q, 1_Q)\).

If \(1_Q T = 1_Q\) for all \(T \in \Theta\) then the following statements are true

(ii) \((\Theta \times Q, \ast)\) is a loop,

(iii) \(N = \{ (\text{id}_Q, x) \mid x \in Q \}\) is a normal subloop of \((\Theta \times Q, \ast)\)

(iv) \(H = \{ (T, 1_Q) \mid T \in \Theta \}\) is a subloop of \((\Theta \times Q, \ast)\),

(v) \(\Theta \times Q = H \ast N\) and \(H \cap N = \{ (\text{id}_Q, 1_Q) \}\).

Denoting the quasigroup \((\Theta \times Q, \ast)\) by \(\text{Hol}_\Theta(Q)\) for any group \(G\) one has \(\text{Hol}(G) = \text{Hol}_{\text{Aut}_G}(G)\).
Pseudoautomorphisms

For a loop $Q$ a bijection $A : Q \rightarrow Q$ is called a right pseudo–automorphic if there exists an element $a \in Q$, called a companion of $A$, such that

$$((xy)A)a = ((xA)((yA)a))$$

for all $x, y \in Q$. We denote by $\text{PsAut}(Q)$ the set of all pseudo–automorphic mappings of a loop $Q$. For $A \in \text{PsAut}(Q)$ we put

$$C(A) = \{a \in Q \mid a \text{ is a companion of } A\}.$$

**Proposition**

*For any loop $Q$ the inclusion $\mathcal{I}_r(Q) \subseteq \text{PsAut}(Q)$ holds.*
Proposition

Let $Q$ be a loop. Then

1. Every automorphism of $Q$ is a pseudo–automorphic mapping of $Q$.
2. If $A \in \text{PsAut}(Q)$, then $1_Q A = 1_Q$.
3. For $A, B \in \text{PsAut}(Q), a \in C(A), b \in C(B)$ one has $AB \in \text{PsAut}(Q)$ and $aB \cdot b \in C(AB)$
4. If $A \in \text{PsAut}(Q)$ and $a \in C(A)$, then $A^{-1} \in \text{PsAut}(Q)$ has a companion $c$ for which $aA^{-1} \cdot c = 1_Q$. 
Assume: $Q$ a loop, $A \in \text{PsAut}(Q)$, $a \in \mathcal{C}(A)$. One calls $(A, a)$ an extended pseudo–automorphism of $Q$ and denote by $\text{EPsAut}(Q)$ the set of all pseudo–automorphisms of $Q$.

For $(A, a), (B, b)$ one defines

$$(A, a) \circ (B, b) = (A \circ B, (Ba)b).$$

From Proposition 3 follows

**Theorem**

*If $Q$ is loop, then $(\text{EPsAut}(Q), \circ)$ is a group.*

Note that for a group $G$ one has $\text{EPsAut}(G) = \text{Aut}(G) \ltimes G$. 
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\[(A, x) \ast (B, y) = (AB, xB \cdot y) \tag{3}\]
a multiplication is defined on the set \(\Theta \times Q\). One has:
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Denoting the quasigroup \((\Theta \times Q, \ast)\) by \(\text{Hol}_\Theta(Q)\) for any group \(G\) one has \(\text{Hol}(G) = \text{Hol}_{\text{Aut}_G}(G)\).
The Sabinin product


In 1972 Sabinin described a construction of the group $\text{Mult}(Q)$ from a given loop structure on the set $Q$ and the group $I_r(Q)$.

Given a loop $(Q, \cdot)$ and a subgroup $\Theta \leq \text{Sym}_0(Q)$, the stabilezer of $1_Q$ in $\text{Sym}(Q)$ we consider the set $S = \Theta \times Q$ and the injections and projections

\[
\begin{align*}
\nu_1 : \Theta & \rightarrow S, \; \vartheta \mapsto (\vartheta, 1_Q), \\
\pi_1 : S & \rightarrow \Theta, \; (\vartheta, x) \mapsto \vartheta, \\
\nu_2 : Q & \rightarrow S, \; x \mapsto (\text{id}_Q, x), \\
\pi_2 : S & \rightarrow Q, \; (\vartheta, x) \mapsto x.
\end{align*}
\]
The Sabinin product/2

Definition
For an arbitrary mapping \( \varphi : (\Theta \times Q) \times (\Theta \times Q) \to \Theta \) define on
\[ S = \Theta \times Q \] a multiplication \( \star_\varphi \) by

\[
s_1 \star_\varphi s_2 = (\vartheta_1, x_1) \star_\varphi (\vartheta_2, x_2) = (\varphi((\vartheta_1, x_1), (\vartheta_2, x_2)), x_1 \vartheta_2 \cdot x_2),
\]

using the notation \( s_i = (\vartheta_i, q_i) \).

We call this multiplication \( \star_\varphi \) the Sabinin multiplication. One sees that

\[
\pi_2((\vartheta_1, x_1) \star_\varphi (\vartheta_2, x_2)) = x_1 \vartheta_2 \cdot x_2.
\]

for all \((\vartheta_1, x_1), (\vartheta_2, x_2) \in \Theta \times Q\). It follows that in the special case

\[
\varphi((\vartheta_1, x_1), (\vartheta_2, x_2)) = \vartheta_1 \vartheta_2
\]

the Sabinin magma \((\Theta \times Q, \star_\varphi)\) coincides with the loop \(\text{Hol}_\Theta(Q)\)
(Theorem 7).
\[ s_1 \ast_\varphi s_2 = (\vartheta_1, x_1) \ast_\varphi (\vartheta_2, x_2) = (\varphi((\vartheta_1, x_1), (\vartheta_2, x_2)), x_1 \vartheta_2 \cdot x_2) \]

It is not difficult to give conditions that the magma \((\Theta \times Q, \ast_\varphi)\) has a neutral element, associativity is somewhat harder.

**Proposition**

Given \(\Theta, Q, \varphi\) put \(s_i = (\vartheta_i, q_i)\) and \(\varphi(s_i, s_j) = \varphi((\vartheta_i, q_i), (\vartheta_j, q_j))\). Then the multiplication \(\ast_\varphi\) is a associative if and only if the identities

\[
\varphi\left((\varphi(s_1, s_2), q_1 \vartheta_2 \cdot q_2), s_3\right) = \varphi\left(s_1, (\varphi(s_2, s_3), q_2 \vartheta_3 \cdot q_3)\right), \tag{7}
\]

\[
(q_1 \vartheta_2 \cdot q_2)\vartheta_3 \cdot q_3 = q_1 \varphi(s_2, s_3) \cdot (q_2 \vartheta_3 \cdot q_3). \tag{8}
\]
Using Proposition 5 one shows in the special case that $\Theta = \mathcal{I}_r(Q)$ for a suitable

$$\rho : (\mathcal{I}_r(Q) \times Q) \times (\mathcal{I}_r(Q) \times Q) \rightarrow \mathcal{I}_r(Q \times Q)$$

Sabinin’s Theorem:

**Theorem**

*For a loop $Q$ the magma $(\mathcal{I}_r(Q) \times Q, \star_\rho)$ is a group isomorphic to $\text{RMult}(Q)$.***
The story continues (Groups with triality)


Now one had to speak about Moufang loops\(^2\), groups with a triality\(^3\), pseudo–automorphisms, autotopisms ...
Mikheev used the concepts described in this talk to construct for a given Moufang loop \(M\) and its group of pseudo–automorphisms \(\text{EPsAut}(M)\) a mapping

\[\mu : (\text{EPsAut}(M) \times M) \times (\text{EPsAut}(M) \times M) \to \text{EPsAut}(M)\]

such that the group \(G(M) = (\text{EPsAut}(M) \times M, \star_\mu)\) is a group with triality that ”coordinizes” \(M\).

\(^2\)\(z(x(zy)) = ((zx)z)y\)
\(^3\)a group on which the symmetric group \(\Sigma_3\) acts as automorphisms satisfying a particular identity
The story continues,

... but for today we stop here.

Thank you for your patience – see you later