The Sabinin product in loops and quasigroups

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This talk is dedicated to the memory of

Lev Vasil'evich Sabinin

21 June 1932 – 4 June 2004

Summary

1. **Left loops and their multiplication group**
2. **Loops and transitive permutation groups**
3. **The holomorph**
3. **Loops and transitive permutation groups**
4. **Sabinin's product**
5. **Groups with a triality**

These pages are just a guide for a spoken text. From a certain point on the presentation switches from left multiplictions to right multiplications.

Left loops

A **magma**[1] $(M, \circ)$ is a set M with a multiplication $\circ$ and a **left loop** is a universal algebra

$$L = \langle L, \backslash, \circ, \mathbf{1}_r \rangle$$

of type $(2, 0, 1)$ satisfying the identities

$a \circ (a \backslash b) = b$,
$a \backslash (a \circ b) = b$,
$a \circ \mathbf{1}_r = a$.

In a left loop Q the **left translations** defined by $L_a(x) = ax$ are bijections. Call the group $\mathrm{LMult}(Q)$ generated by the set $\mathrm{T}_\ell(Q)$ of all left translations of Q the **left multiplication group** of Q.

Note: $(L_a)^{-1}(x) = a \backslash x$.

[1] The name "groupoid" is taboo.

The associant of a left loop

If Q is a left loop, we have the transitive action of the group $\mathrm{LMult}(Q)$ on the set Q. In Sabinin's terminology the stabilizer $\mathcal{I}_\ell(Q) = \mathrm{Stab}_{\mathrm{LMult}(Q)}(\mathbf{1}_r) = \{X \in \mathrm{LMult}(Q) \mid X(\mathbf{1}_r) = \mathbf{1}_r\}$ is called the **left associant** of Q. Others call $\mathcal{I}_\ell(Q)$ the left inner mapping group of Q.

Note that for any $a \in Q$ one has

$$\mathrm{Stab}_{\mathrm{LMult}(Q)}(a) = L_a \mathrm{Stab}_{\mathrm{LMult}(Q)}(\mathbf{1}_r) L_a^{-1}. \tag{1}$$

Right loops

Given a **right loop** $L = \langle \circ, /, \mathbf{1}_\ell \rangle$ we have

- Right multiplications R_b
- The set of all right translations $\mathsf{T}_r(Q)$
- The right multiplication group $\mathrm{RMult}(Q) = \langle \mathsf{T}_r(Q) \rangle$
- The right associant $\mathcal{I}_r(Q) = \mathrm{S}tab_{\mathrm{RMult}(Q)}(\mathbf{1}_\ell)$

Loops

A **loop** is at the same time a left and a right loop. Note that left and right neutral element coincide. So in a loop $L = \langle \circ, \backslash, /, \mathbf{1} \rangle$ we have

- $\mathrm{T}_\ell(Q), \mathrm{LMult}(Q), \mathcal{I}_\ell(Q)$
- $\mathrm{T}_r(Q), \mathrm{RMult}(Q), \mathcal{I}_r(Q)$
- The multiplication group $\mathrm{Mult}(Q)$
- The associant $\mathcal{I}(Q) = \mathrm{S}tab_{\mathrm{RMult}(Q)}(\mathbf{1})$

etc

For left, right or twosided quasigroup Q in the defintion of the associants one chooses an arbitrary base point $x_0 \in Q$ and observes equation 1.

I will not follow that path in my talk.

Permutation groups

R. Baer, Nets and groups. Trans. Amer. Math. Soc. **46**, 110–141 (1939)

Given a transitive action $\eta : G \times X \to X$ of a group G on a set X put $H = H_{\eta,x_0} = \mathrm{Stab}_G(x_0)$ for an arbitrary base point $x_0 \in X$.

Now identify X with coset space G/H via

$g_1x_0 = g_2x_0 \Leftrightarrow x_0 = g_1^{-1}g_2x_0 \Leftrightarrow g_1^{-1}g_2 \in H \Leftrightarrow g_1H = g_2H$

and choose a coset representative system K of H in G.

We consider triples $\mathcal{B} = \big(G, H, K\big)$ - group, subgroup, transversal – and call them **Baer triples**.

Now one defines a multiplication $\diamond_{\mathcal{B}}$ on the set K by

$$(xH)(yH) = (x \diamond_{\mathcal{B}} y)H.$$

Theorem

[Baer] *For a Baer triple $\mathcal{B}$ the magma $(K, \diamond_{\mathcal{B}})$ is a left loop.*

Baers construction

We denote the left loop defined in the last theorem by $Q_{\mathcal{B}}$. Note that we are not planning to discuss its dependence on the transversal K. However, there are some useful observations.

(B.1) $\bigcap_{g\in G} H^g = 1$ if and only if the action of G is faithful.

(B.2) In $Q_{\mathcal{B}}$ there is a twosided neutral element if $1_G \in K$.

(B.3) $Q_{\mathcal{B}}$ is a loop if and only if K is a transversal of H^g for all $g \in G$.

(B.4) $G = \mathrm{LMult}(Q_{\mathcal{B}})$ if and only if K is a generating set of the group G.

Theorem

[BAER] *For a left loop Q and the Baer triple* $\mathcal{B} = (\mathrm{LMult}(Q), \mathcal{I}_\ell(Q), \mathrm{T}_\ell(Q))$ *the left loop $Q_{\mathcal{B}}$ isomorphic to Q.*

The torsor of a group

H. Prüfer, Theorie der Abelschen Gruppen, Math. Zeit. **20**, 166–187 (1924).
R. Baer, Zur Einführung des Scharbegriffs. J. Reine Angew. Mathematatik, **(160**, 199–207 (1929).

W. Bertram, M. Kinyon, Associative geometries. J. Lie Theory **20**, no. 2, 215–252 (2010)

For group G one calls the ternary operation

$$\tau_G(a, b, c) = ab^{-1}c$$

the **torsor** of G. One easily sees

Proposition

If X is a coset of some subgroup of a group G, then $\tau_G(X, X, X) \subseteq X$.

An origin of the notion of the torsor lies in affine geometry.

The holomorph of a group

Let G be a group. Then a bijection $\beta : G \to G$ is called a **holomorphism** if $\tau_G(a^\beta, b^\beta, c^\beta) = \tau_G(a, b, c)^\beta$ for all $a, b, c \in G$. The set $\mathrm{Hol}(G)$ of all holomorphisms of G forms a group, the **holomorph** of G.

Theorem

For any group G one has $\mathrm{LMult}(G), \mathrm{RMult}(G) \triangleleft \mathrm{Hol}(G)$ *and* $\mathrm{Aut}(G) \leq \mathrm{Hol}(G)$. *Furthermore,* $\mathrm{Inn}(G) \leq \mathrm{Mult}(G)$ *and*

$$\mathrm{Hol}(G) = \mathrm{Aut}(G) \ltimes \mathrm{LMult}(G) = \mathrm{Aut}(G) \ltimes \mathrm{RMult}(G) \cong \mathrm{Aut}(G) \ltimes G.$$

The holomorph of a loop

Theorem

For a loop $(Q, \cdot)$ and a group Θ acting – not necessarily faithfully – as a group of permutations on the set Q by

$$(A, x) * (B, y) = (AB, xB \cdot y) \tag{2}$$

a multiplication is defined on the set $\Theta \times Q$. One has:
(i) $(\Theta \times Q, *)$ is a quasigroup with the left neutral element $(\mathrm{id}_Q, 1_Q)$.

If $1_Q T = 1_Q$ for all $T \in \Theta$ then the following statements are true

(ii) $(\Theta \times Q, *)$ is a loop,
(iii) $N = \{(\mathrm{id}_Q, x) \mid x \in Q\}$ is a normal subloop of $(\Theta \times Q, *)$
(iv) $H = \{(T, 1_Q) \mid T \in \Theta\}$ is a subloop of $(\Theta \times Q, *)$,
(v) $\Theta \times Q = H * N$ and $H \cap N = \{(\mathrm{id}_Q, 1_Q)\}$.

Denoting the quasigroup $(\Theta \times Q, *)$ by $\mathrm{Hol}_\Theta(Q)$ for any group G one has $\mathrm{Hol}(G) = \mathrm{Hol}_{\mathrm{Aut}\,G}(G)$.

Pseudoautomorphisms

For a loop Q a bijection $A : Q \to Q$ is called a *right pseudo–automorphic* if there exists an element $a \in Q$, called a *companion* of A, such that

$$((xy)A)a = ((xA)((yA)a))$$

for all $x, y \in Q$. We denote by $\mathrm{PsAut}(Q)$ the set of all pseudo–automorphic mappings of a loop Q. For $A \in \mathrm{PsAut}(Q)$ we put

$$\mathcal{C}(A) = \{a \in Q \mid \ a \text{ is a companion of } A\}.$$

Proposition

For any loop Q the inclusion $\mathcal{I}_r(Q) \subseteq \mathrm{PsAut}(Q)$ holds.

Pseudoautomorphisms/2

$$((xA)(yA)a) = ((xy)A)a$$

Proposition

Let Q be a loop. Then

(1) Every automorphism of Q is a pseudo–automorphic mapping of Q.

(2) If $A \in \mathrm{PsAut}(Q)$, then $1_Q A = 1_Q$.

(3) For $A, B \in \mathrm{PsAut}(Q), a \in \mathcal{C}(A), b \in \mathcal{C}(b)$ one has $AB \in \mathrm{PsAut}(Q)$ and $aB \cdot b \in \mathcal{C}(AB)$

(4) If $A \in \mathrm{PsAut}(Q)$ and $a \in \mathcal{C}(A)$, then $A^{-1} \in \mathrm{PsAut}(Q)$ has a companion c for which $aA^{-1} \cdot c = 1_Q$.

Pseudoautomorphisms/3

$$((xA)(yA)a) = ((xy)A)a$$

Assume: Q a loop, $A \in \mathrm{PsAut}(Q)$, $a \in \mathcal{C}(A)$. One calls (A, a) an **extended pseudo–automorphism** of Q and denote by $\mathrm{EPsAut}(Q)$ the set of all pseudo–automorphisms of Q.

For $(A, a), (B, b)$ one defines

$$(A, a) \circ (B, b) = (A \circ B, (Ba)b).$$

From Proposition 3 follows

Theorem

If Q is loop, then $(\mathrm{EPsAut}(Q), \circ)$ is a group.

Note that for a group G one has $\mathrm{EPsAut}(G) = \mathrm{Aut}(G) \ltimes G$.

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R. Baer, Zur Einführung des Scharbegriffs. J. Reine Angew. Mathematik, **(160**, 199–207 (1929).

R. B. Bruck, L. J. Paige, Loops whose inner mappings are automorphisms. Ann. Math. **63**, 308–323 (1956).

W. Bertram, M. Kinyon, Associative geometries. I: torsors, linear relations and Grassmannians. J. Lie Theory **20**, no. 2, 215–252 (2010)

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The Sabinin product

L. V. Sabinin, On the equivalence of category of loops and the category of homogeneous spaces. (Russian) Dokl. Akad. Nauk SSSR **205**, no. 4, 970–974 (1972); translation in Soviet Math. Dokl. **13**, no. 4, 970–974 (1972)

P. O. Mikheev, L. V. Sabinin, *Quasigroups and differential geometry*. Quasigroups and loops: theory and applications, pp. 357–430, Sigma Ser. Pure Math., 8, Heldermann, Berlin, 1990. [Chapter 12]

L. V. Sabinin, Smooth quasigroups and loops. Mathematics and its Applications, Kluwer Academic Publishers 1999.

In 1972 Sabinin described a construction of the group $\mathrm{Mult}(Q)$ from a given loop structure on the set Q and the group $\mathcal{I}_r(Q)$.

Given a loop $(Q, \cdot)$ and a subgroup $\Theta \leq \mathrm{Sym}_0(Q)$, the stabilezer of 1_Q in $\mathrm{Sym}(Q)$ we consider the set $S = \Theta \times Q$ and the injections and projections

$$\iota_1 : \Theta \to S,\ \vartheta \mapsto (\vartheta, 1_Q), \qquad \iota_2 : Q \to S,\ x \mapsto (\mathrm{id}_Q, x)$$
$$\pi_1 : S \to \Theta,\ (\vartheta, x) \mapsto \vartheta, \qquad \pi_2 : S \to Q,\ (\vartheta, x) \mapsto x.$$

The Sabinin product/2

Definition

For an arbitrary mapping $\varphi : (\Theta \times Q) \times (\Theta \times Q) \to \Theta$ define on $S = \Theta \times Q$ a multiplication $\star_\varphi$ by

$$s_1 \star_\varphi s_2 = (\vartheta_1, x_1) \star_\varphi (\vartheta_2, x_2) = (\varphi\big((\vartheta_1, x_1), (\vartheta_2, x_2)\big), x_1\vartheta_2 \cdot x_2), \qquad (4)$$

using the notation $s_i = (\vartheta_i, q_i)$.

We call this multipliction $\star_\varphi$ the *Sabinin multiplication*. One sees that

$$\pi_2\big((\vartheta_1, x_1) \star_\varphi (\vartheta_2, x_2)\big) = x_1\vartheta_2 \cdot x_2. \qquad (5)$$

for all $(\vartheta_1, x_1), (\vartheta_2, x_2) \in \Theta \times Q$. It follows that in the special case

$$\varphi\big((\vartheta_1, x_1), (\vartheta_2, x_2)\big) = \vartheta_1\vartheta_2 \qquad (6)$$

the Sabinin magma $(\Theta \times Q, \star_\varphi)$ coincides with the loop $\mathrm{Hol}_\Theta(Q)$ (Theorem 7).

The Sabinin product/3

$$s_1 \star_\varphi s_2 = (\vartheta_1, x_1) \star_\varphi (\vartheta_2, x_2) = \left(\varphi\big((\vartheta_1, x_1), (\vartheta_2, x_2)\big), x_1\vartheta_2 \cdot x_2\right)$$

It is not difficult to give conditions that the magma $(\Theta \times Q, \star_\varphi)$ has a neutral element, associativity is somewhat harder.

Proposition

Given Θ, Q, φ put $s_i = (\vartheta_i, q_i)$ and $\varphi(s_i, s_j) = \varphi\big((\vartheta_i, q_i), (\vartheta_j, q_j)\big)$. Then the multipliction $\star_\varphi$ is a associative if and only if the identities

$$\varphi\Big(\big(\varphi(s_1, s_2), q_1\vartheta_2 \cdot q_2\big), s_3\Big) = \varphi\Big(s_1, \big(\varphi(s_2, s_3), q_2\vartheta_3 \cdot q_3\big)\Big), \qquad (7)$$

$$(q_1\vartheta_2 \cdot q_2)\vartheta_3 \cdot q_3 = q_1\varphi(s_2, s_3) \cdot (q_2\vartheta_3 \cdot q_3). \qquad (8)$$

Sabinins Theorem

Using Proposition 5 one shows in the special case that $\Theta = \mathcal{I}_r(Q)$ for a suitable

$$\rho : (\mathcal{I}_r(Q) \times Q) \times (\mathcal{I}_r(Q) \times Q) \to \mathcal{I}_r(Q \times Q)$$

Sabinin's Theorem:

Theorem

For a loop Q the magma $(\mathcal{I}_r(Q) \times Q, \star_\rho)$ is a group isomorphic to $\mathrm{RMult}(Q)$.

The story continues (Groups with triality)

S. DORO, Simple Moufang loops. Math. Proc. Cambridge Philos. Soc. 83, no. 3, 377–392 (1978)

P. O. MIKHEEV, Moufang loops and their enveloping groups. Webs and quasigroups, pp. 33–43, Tver. Gos. Univ. (1993)

J. I. HALL, On Mikheev's construction of enveloping groups. Comment. Math. Univ. Carolin. **51**, no. 2, 245–252 (2010)

Now one had to speak about Moufang loops[2], groups with a triality[3], pseudo–automorphisms, autotopisms ...
Mikheev used the concepts described in this talk to construct for a given Moufang loop M and its group ot pseudo–automorphisms $\mathrm{EPsAut}(M)$ a mapping

$$\mu : (\mathrm{EPsAut}(M) \times M) \times (\mathrm{EPsAut}(M) \times M) :\to \mathrm{EPsAut}(M)$$

such that the group $\mathcal{G}(M) = (\mathrm{EPsAut}(M) \times M, \star_\mu)$ is a group with triality that "coordinizes" M.

[2] $z(x(zy)) = ((zx)z)y$

[3] a group on which the symmetric group Σ_3 acts as automorphisms satisfying a particular identity

The story continues,

... but for today we stop here.

Thank you for your patience – see you later