A dense family of finite 1-generated left-distributive groupoids

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Left-Distributive Groupoids

A groupoid $\boldsymbol{G} = \langle G; *\rangle$ is (left)-distributive if

$$\boldsymbol{G} \models \forall xyz\ x * (y * z) = (x * y) * (x * z)$$

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In this talk, we will be interested in monogenerated LD groupoids (MLDs). For example, since $F \Rightarrow F = T$, the example above is generated by F (but not by T).

Nonidempotent LD groupoids I

LD groupoids encountered in the wild (i.e. in knot theory) are frequently idempotent:

$$\forall x \; x * x = x$$

Idempotent 1-generated groupoids, however, are boring, so we'll ignore them.
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Theorem (Dehornoy 1992)

1. *(ZFC) There exists a concrete representation of* $\boldsymbol{F}_{\mathcal{LD}}(1)$ *by* B_ω *(the group of braids on finitely many strands).*
2. *(ZFC) The coloring of* B_ω *above induces a linear ordering compatible with the group multiplication.*

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Theorem (Laver 1989)

If a certain large cardinal axiom holds, then

1. *there exists a "concrete" representation of* $\boldsymbol{F}_{\mathcal{LD}}(1)$ *as a set of elementary embeddings;*
2. $\boldsymbol{F}_{\mathcal{LD}}(1)$ *is residually finite.*

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Open Problem

(ZFC) Show that $\boldsymbol{F}_{\mathcal{LD}}(1)$ *is residually finite.*

Finite Quotients

Laver actually showed more: he exhibited a set of finite groupoids $\{\boldsymbol{LT}_n : n \geq 0\}$ of cardinality 2^n, such that $\boldsymbol{F}_{\mathcal{LD}}(1)$ is residually $\{\boldsymbol{LT}_n\}$ under the same large cardinal assumptions. These groupoids (called Laver Tables) generalize the example of $\langle\{T, F\}; \Rightarrow\rangle$ given above.

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Open Problem

Call the stronger residual statement above (L).

- *(Optimist's version) (L) is a theorem of ZFC*
- *(Cautious Optimist's version) Residual finiteness of* $\boldsymbol{F}_{\mathcal{LD}}(1)$ *is a theorem of ZFC*
- *(Pessimist's version) (L) is not provable in ZFC alone*
- *(Ultrapessimist's version)* $\neg$*(L) is a theorem of ZFC*

Slender MLD groupoids

We say $\boldsymbol{G}$ has *Laver dimension n* if

$$\boldsymbol{G} \overset{\pi}{\twoheadrightarrow} \boldsymbol{LT}_n \quad \text{but} \quad \boldsymbol{G} \not\twoheadrightarrow \boldsymbol{LT}_{n+1}$$

and is *slender* if

$$a \equiv_\pi b \quad \Rightarrow \quad \forall x \; a * x = b * x$$

Fact

- *If terms $t_1(x), t_2(x)$ have different right branch depths, then there exists a finite zero-dimensional MLD in which they evaluate differently.*
- *If $\boldsymbol{LT}_n \models t_1(x) = t_2(x)$ and the terms' right branch depths are equal, then $\boldsymbol{G} \models t_1(x) = t_2(x)$ for every finite slender n-dimensional MLD $\boldsymbol{G}$.*

Isomorphism Classification – Slender Case

Theorem (Many authors, see S. 2013)

- *The family* $\{\mathbf{LT}_n : n \geq 0\}$ *is a chain with respect to the homomorphism order (in particular, this family is inverse directed).*
- *The family of (finite) slender MLDs is classified up to isomorphism by* n *and two function parameters* $\rho, \nu : 2^n \to \omega$*, which can be chosen independently of each other.*
- *Slender MLDs admit a dense subfamily parametrized by integers* $n \geq 0, r \geq 1, v \geq 0$*, inverse directed by the usual ordering in* n, v *and by divisibility in* r*.*

Isomorphism Classification – Nonslender Case

Theorem (Drápal 1997)

The family of all finite MLDs is classified up to isomorphism by n and seven function parameters.

This classification is great for theory but of little practical use on its own, since the parameters are highly interdependent. (The full statement of the classification theorem takes about a page.) Virtually every author discussing MLD groupoids restricts most of their attention to the slender case; the nonslender family's Homeric epithet is "combinatorially chaotic".

Main Theorem

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Theorem (S.)

There exists a family

$$\mathcal{F} = \{\boldsymbol{F}(n, r, v, w_1, w_2) : n \geq 0, r \geq 1, v \geq 2, w_1 \geq 0, w_2 \geq 1\}$$

of finite MLD groupoids, such that

- *Every finite MLD groupoid* $\boldsymbol{G}$ *is a quotient of a member of* $\mathcal{F}$*, and finding one which does so is tractably computable from the multiplication table of* $\boldsymbol{G}$*;*
- $\mathcal{F}$ *is inverse-directed by the usual ordering on* n, v, w_1 *and by divisibility in* r, w_2*.*

Well-behaved?

I refer to the groupoids $\mathcal{F}$ as "well-behaved" for a couple of reasons:

- The five parameters are integers and can be chosen independently of each other.
- Each of the seven function parameters in Drápal's classification is chosen in "the most natural possible" way, to identify as few elements as possible.
- $\mathcal{F}$ is inverse-directed, and it is easy to determine whether one member of $\mathcal{F}$ is a quotient of another.
- $\mathcal{F}$ "automatically" separates all terms of different right branch depths.

Well-behaved?

The “combinatorial chaos” in $\mathcal{LD}$ involves basically terms of right branch depth 1 and 2. One way of thinking about the groupoids $\mathcal{F}$ is to take a slender groupoid with $v \geq 2$ and split some of its elements, obtained from the generator by terms of right branch depth 1 or 2, up into pieces in a uniform way.

Room for cautious optimism

Open Problem (ZFC)

Is $\boldsymbol{F}_{\mathcal{LD}}(1)$ residually finite?

Example (Dougherty & Jech)

The function

$$f(m) = \min\{n : \boldsymbol{LT}_n \models 1 * 1 \neq 1 * 1_{[2^m+1]}\}$$

grows faster than any primitive recursive function. For example, when $m = 4$, $f(m) \geq \mathrm{Ack}(9, \mathrm{Ack}(8, \mathrm{Ack}(8, 254)))$

However, these two terms are clearly not LD-equivalent (they have different right branch lengths).

Room for cautious optimism

Example

Let

$$t_1(x) = x_{[5]} * (x_{[2]} * x) \quad t_2(x) = x * ((x * x_{[3]}) * x)$$

We have

$$\boldsymbol{LT}_2 \models t_1 \approx t_2 \quad \text{and } \mathrm{d}_r(t_1) = \mathrm{d}_r(t_2) = 2$$

Hence t_1 and t_2 evaluate identically in every slender MLD groupoid of dimension 2. However we have

$$\boldsymbol{F}(2, 3, 2, 0, 1) \models t_1 \not\approx t_2$$

Problems

The groupoids $\mathcal{F}$ provide some level of control or upper bound on the combinatorial explosion present in terms of right branch length ≤ 2.

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Open Problem

- *Use $\mathcal{F}$ to improve Dehornoy's normal form result for LD terms in one variable.*

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Open Problem

- *Use $\mathcal{F}$ to prove residual finiteness of $\mathbf{F}_{\mathcal{LD}}(1)$.*

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Open Problem

- *Inverse limits in $\mathcal{F}$, where at least one of the five parameters is bounded, should provide many new examples of infinite nonfree LD groupoids. Do these groupoids represent naturally (e.g. as injection brackets [Dehornoy 2000]) on familiar spaces?*

Problems

Open Problem

Does there exist a (manageable, useful) presentation for $\boldsymbol{F}(n, r, v, w1, w2)$ *by generator and relations? (Drápal showed the existence of such a presentation for the slender MLD groupoids; we have some idea what this would need to look like, but no complete description even for the smallest actual examples.)*

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It is known that the equational theory of the variety $\mathcal{LD}$ is decidable (this is the same thing as the word problem for the free algebras $\mathbf{F}_{\mathcal{LD}}(n)$);

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1. *is the first-order theory of $\mathcal{LD}$ decidable?*

Problems

Open Problem

It is known that the equational theory of the variety $\mathcal{LD}$ is decidable (this is the same thing as the word problem for the free algebras $\mathbf{F}_{\mathcal{LD}}(n)$);

2. *is $\mathcal{LD}$ axiomatized by the equations which hold in $\mathbf{F}_{\mathcal{LD}}(1)$?*

Problems

Open Problem

It is known that the equational theory of the variety $\mathcal{LD}$ is decidable (this is the same thing as the word problem for the free algebras $\mathbf{F}_{\mathcal{LD}}(n)$);

3. *does there exist a first-order formula which does not hold throughout $\mathcal{LD}$, but which does hold in every finite LD groupoid?*

Bibliography

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