Quasigroup Actions and Approximate Symmetry

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Recall that $Px = Py$ or $Px \cap Py = \emptyset$ in a group Q.

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has a $|P\backslash Q| \times |P\backslash Q|$ row-stochastic

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$$[R_{P\backslash Q}(q)]_{XY} = |Xq \cap Y| \big/ |X|$$

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Have dual versions $\quad \mathrm{RMlt}_Q P, \quad Q/P, \quad L_{Q/P}(q) = |qX \cap Y| \big/ |X|, \quad \ldots$

Agenda

1. Lagrangian properties.
2. Burnside's Lemma.
3. Sylow theory.
4. A simple Bol loop acting on a projective line.
5. Approximately symmetric fractal-type objects.

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On the other hand, the empty subquasigroup is both right and left Lagrangean.

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E.g: $$R_{\langle (1\ 2) \rangle \backslash \{0,1,2\}!}\big((0\ 2)\big) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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To extend Burnside's Lemma to quasigroup actions, prove $(*)$.

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Lemma: For $q \in Q$, have $R_{P\backslash Q}(q) = A_P^+ R_Q(q) A_P$,

where $R_Q(q)$ is the permutation matrix of $R(q)$ on Q.

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&= \frac{1}{|Q|}\sum_{X\in P\backslash Q}\sum_{x\in X} |X|^{-1}|X| \qquad = \frac{1}{|Q|}\sum_{x\in Q} 1 = 1
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Sylow's Theorem (part): If d is a prime power divisor of $|Q|$ for a finite group Q,

then good orbits exist, and each contains a (Lagrangean) subquasigroup.

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Containments:

$$\begin{tikzcd}
\mathrm{G}^* \arrow[r] & \mathrm{H}^* \arrow[r] & \mathrm{I}^* \arrow[r] & \mathrm{J}^* \\
\mathrm{G} \arrow[r] \arrow[ur, "\backslash" marking] & \mathrm{H} \arrow[r] & \mathrm{I} \arrow[r] \arrow[u] & \mathrm{J} \arrow[u]
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while Klein 4-subgroups "of negative type" have orbits of length $120 - 6$.

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4. A simple BoI loop acting on a projective line.

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along with successive conjugates $0_1, 0_2, 1_1, 1_2, 2_1, 2_2, 3_1, 3_2, 4_1, 4_2$ by the shift.

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Define $N = N_\infty \dot\cup N_0 \dot\cup N_1 \dot\cup N_2 \dot\cup N_3 \dot\cup N_4$, a simple right Bol loop of order $96 = 16 \cdot 6$.

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The right and left homogeneous spaces of the nub take the form

$$N_\infty \backslash N = N/N_\infty = \{N_\infty, N_0, N_1, N_2, N_3, N_4\}\,,$$

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For $x \in \mathsf{GF}(5)$ and $d \in \{1, 2\}$,

each element $p \cdot x_d$ of $N_{x,d}$ (with p in ∞_x)

acts on $\mathsf{PG}_1(5)$ as the permutation x_d.

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For $p \in \infty_0$ and $d \in \{1, 2\}$,

$$L_{N/N_\infty}(p \cdot 0_d) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

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Other matrices obtained on conjugation by the shift.

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For $p \in \infty_0$ and $d \in \{1, 2\}$,

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Agenda

5. Approximately symmetric fractal-type objects.

Quasigroup actions as iterated function systems

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consider the simplex $(P\backslash Q)^B$ spanned by the points of $P\backslash Q$ as a compact metric space.

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Consider

$$\bigcup_{x\in P\backslash Q} \bigcup_{n\in\mathbb{N}} \bigcup_{q_i\in Q} xR_{P\backslash Q}(q_1)\dots R_{P\backslash Q}(q_i)\dots R_{P\backslash Q}(q_n)$$

as an affine geometric subset of the simplex $(P\backslash Q)^B$.

Chalmers' example

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Consider the subquasigroup $P = \{1\}$ in

Q	1	2	3	4	5	6
1	1	3	2	5	4	6
2	2	4	5	1	6	3
3	3	5	6	4	1	2
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Have $P\backslash Q = \big\{\{1\}, \{2,3\}, \{4,5,6\}\big\}$, so $(P\backslash Q)^B$ is a triangle (2-dimensional simplex).