

Quasigroup Actions and Approximate Symmetry

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Recall that $Px = Py$ or $Px \cap Py = \emptyset$ in a group Q .

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has a $|P \setminus Q| \times |P \setminus Q|$ row-stochastic

(right) action matrix $R_{P \setminus Q}(q)$ with (X, Y) -entry

$$[R_{P \setminus Q}(q)]_{XY} = |Xq \cap Y|/|X|$$

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Have dual versions $\text{RMlt}_Q P, Q/P, L_{Q/P}(q) = |qX \cap Y|/|X|, \dots$

Agenda

1. Lagrangian properties.
2. Burnside's Lemma.
3. Sylow theory.
4. A simple Bol loop acting on a projective line.
5. Approximately symmetric fractal-type objects.

Agenda

1. Lagrangian properties.

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On the other hand, the empty subquasigroup is both right and left Lagrangean.

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To extend Burnside's Lemma to quasigroup actions, prove (*).

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Pseudoinverse A^+ with $A_{Xx}^+ = \begin{cases} |X|^{-1} & \text{if } x \in X; \\ 0 & \text{otherwise} \end{cases} \quad \text{for } X \in P \setminus Q \text{ and } x \in Q.$

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Lemma: For $q \in Q$, have $R_{P \setminus Q}(q) = A_P^+ R_Q(q) A_P$,

where $R_Q(q)$ is the permutation matrix of $R(q)$ on Q .

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For a divisor d of the order of a finite quasigroup Q ,

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Sylow's Theorem (part): If d is a prime power divisor of $|Q|$ for a finite group Q ,
then good orbits exist, and each contains a (Lagrangean) subquasigroup.

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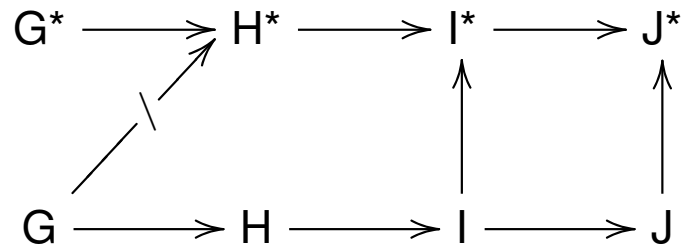
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Containments:



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while Klein 4-subgroups “of negative type” have orbits of length $120 - 6$.

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4. A simple Bol loop acting on a projective line.

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along with successive conjugates $0_1, 0_2, 1_1, 1_2, 2_1, 2_2, 3_1, 3_2, 4_1, 4_2$ by the shift.

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Define $N = N_\infty \dot{\cup} N_0 \dot{\cup} N_1 \dot{\cup} N_2 \dot{\cup} N_3 \dot{\cup} N_4$, a simple right Bol loop of order $96 = 16 \cdot 6$.

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The right and left homogeneous spaces of the nub take the form

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For $x \in \text{GF}(5)$ and $d \in \{1, 2\}$,

each element $p \cdot x_d$ of $N_{x,d}$ (with p in ∞_x)

acts on $\text{PG}_1(5)$ as the permutation x_d .

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For $p \in \infty_0$ and $d \in \{1, 2\}$,

$$L_{N/N_\infty}(p \cdot 0_d) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

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Other matrices obtained on conjugation by the shift.

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For $p \in \infty_0$ and $d \in \{1, 2\}$,

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Agenda

5. Approximately symmetric fractal-type objects.

Quasigroup actions as iterated function systems

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Consider

$$\bigcup_{x \in P \setminus Q} \bigcup_{n \in \mathbb{N}} \bigcup_{q_i \in Q} x R_{P \setminus Q}(q_1) \dots R_{P \setminus Q}(q_i) \dots R_{P \setminus Q}(q_n)$$

as an affine geometric subset of the simplex $(P \setminus Q)^B$.

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Consider the subquasigroup $P = \{1\}$ in

Q	1	2	3	4	5	6
1	1	3	2	5	4	6
2	2	4	5	1	6	3
3	3	5	6	4	1	2
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6	6	2	1	3	5	4

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Have $P \setminus Q = \{1\}, \{2, 3\}, \{4, 5, 6\}$, so $(P \setminus Q)^B$ is a triangle (2-dimensional simplex).

