Symmetry in the theory of quasigroup and isotopy-isomorphy problem

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Invertible operations and their parastrophes

Let $S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, where $s := (12)$, $\ell := (13)$, $r := (23)$, be the group of all permutations of $\{1, 2, 3\}$.

Parastrophes

Let $(Q; \cdot)$ be a quasigroup, then $\{ \overset{\sigma}{\cdot} \mid \sigma \in S_3\}$ is the set of all parastrophes of the invertible operation $(\cdot)$, where

$$x_{1\sigma} \overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma} \iff x_1 \cdot x_2 = x_3, \quad \sigma \in S_3.$$

Parastrophes of objects

Let P be an arbitrary object of the quasigroup theory, i.e. theorem, lemma, notion,… An object, being obtained from P by replacing $(\overset{\tau}{\cdot})$ with $(\overset{\tau\sigma^{-1}}{\cdot})$ for all $\tau \in S_3$, will be called *σ-parastrophe* of P and will be denoted by ${}^{\sigma}P$.

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Actions of S_3

A mapping $(\sigma; P) \mapsto {}^{\sigma}P$ is an action of S_3 on $\{{}^{\sigma}P \mid \sigma \in S_3\}$.

$\mathrm{Sym}(P) := \{\sigma \mid {}^{\sigma}P = P\} \leqslant S_3$ is *symmetry group* of P.
A number of different parastrophes of P is $6/|\mathrm{Sym}(P)|$.
Symmetry groups of parastrophic objects are isomorphic.

So, we can classify the objects according to their symmetry groups: An object P is called:

	totally symmetric,	if	$\mathrm{Sym}(P) = S_3$
	skew symmetric,	if	$\mathrm{Sym}(P) = A_3$
One-side symmetric	*left symmetric,*	if	$\mathrm{Sym}(P) = \{\iota, \ell\}$
	right symmetric,	if	$\mathrm{Sym}(P) = \{\iota, r\}$
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For quasigroups we save the traditional names.

An quasigroup $(Q;\cdot)$ is called:

	totally symmetric,	if	$\mathrm{Sym}(Q;\cdot) = S_3$;	$xy = yx$, $x \cdot xy = y$;
	skew symmetric,	if	$\mathrm{Sym}(Q;\cdot) \supseteq A_3$;	$x \cdot yx = y$;
One-side symmetric	*commutative*,	if	$\mathrm{Sym}(Q;\cdot) \supseteq \{\iota, s\}$;	$xy = yx$;
	left symmetric,	if	$\mathrm{Sym}(Q;\cdot) \supseteq \{\iota, r\}$;	$x \cdot xy = x$;
	right symmetric,	if	$\mathrm{Sym}(Q;\cdot) \supseteq \{\iota, \ell\}$;	$xy \cdot y = y$;
	asymmetric,	if	$\mathrm{Sym}(Q;\cdot) = \{\iota\}$.	

Classes of quasigroups 1

For example, let $\mathfrak{A}$ be a class of quasigroups and then ${}^{\sigma}\mathfrak{A}$ denotes the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$.

Totally symmetric classes of quasigroups

1. the class of all quasigroups;
2. the class of all totally symmetric quasigroups;
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6. the class of all self-orthogonal quasigroups;
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One-side symmetric classes of quasigroups

1. the classes of all loops, groups, Moufang loops, . . . ;
2. the classes of all quasigroups satisfying the 2-nd Stein law;
3. the classes of all quasigroups satisfying the 3-d Stein law.

Asymmetric classes of quasigroups

1. the class of all left (right) loops;
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No variety of skew symmetric class of quasigroups is found.

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Hexality rule or sixfold symmetry

Theorem. Let P be an arbitrary proposition, then P is true in $\mathfrak{A}$ if and only if and only if ${}^{\sigma}P$ is true in ${}^{\sigma}\mathfrak{A}$.

Corollary 1. Let P be true in a class of quasigroups $\mathfrak{A}$, then for all $\sigma \in \mathrm{Sym}(\mathfrak{A})$ ${}^{\sigma}P$ is true in $\mathfrak{A}$.

Corollary 2. Let P be true in a totally symmetric class $\mathfrak{A}$, then for all $\sigma \in \mathrm{Sym}(\mathfrak{A})$ ${}^{\sigma}P$ is true in $\mathfrak{A}$.

Conclusion. Introducing a notion P we have to introduce six pairwise parastrofic notions $\{{}^{\tau}P \mid \tau \in S_3\}$ simultaneously.

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Parastrophic closure of translations

Left, right and middle translations are defined by

$$L_a(x) := a \cdot x, \quad R_a(x) := x \cdot a, \quad M_a(x) = y \;:\Leftrightarrow\; x \cdot y = a.$$

The notion of “translation” is asymmetric:

$$\{^{(\iota)}L_a;\ ^{(\ell)}L_a;\ ^{(r)}L_a;\ ^{(s)}L_a;\ ^{(s\ell)}L_a;\ ^{(sr)}L_a\} =$$

$$= \{L_a;\ M_a^{-1},\ L_a^{-1};\ R_a;\ R_a^{-1};\ M_a\}.$$

Left, right and middle neutral elements are defined by

$$e_\ell \cdot x := x, \qquad x \cdot e_r := x, \qquad x \cdot x := e_m.$$

These notions are one-side symmetric:

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There are seven kinds of loops

"One-side loop" is two-side symmetric:

left loop (with e_ℓ); right loop (with e_r); middle loop (with e_m).

"Two-side loop" is two-side symmetric:

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Symmetric statments

A well-known proposition:

Every quasigroup is isotopic to a loop (=left-right loop):

$$x \circ y = R_a^{-1}(x) \cdot L_b^{-1}(y), \qquad e_\ell = e_r = ba = R_a(b) = L_b(a);$$

Its symmetric propositions:

Every quasigroup is isotopic to a right-midle loop:

$$x \circ y = R_b^{-1}\Big(x \cdot M_c(y)\Big), \qquad e_r = e_m = R_b^{-1}(c) = M_c^{-1}(b);$$

Every quasigroup is isotopic to a left-middle loop:

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Let an identity $\omega = \upsilon$ define a variety $\mathfrak{A}$, then its σ-parastrophic identity defines the variety ${}^{\sigma}\mathfrak{A}$. Two identities are said to be:

- *equivalent*, if they define the same variety;
- *parastrophic*, if they are σ-parastrophic for some σ;
- *isoparastrophic*, if they define parastrophic varieties.

Problem: ***Describe identities up to isoparastrophy.***

Parastrophic varieties

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Theorem. [V.D.Belousov, 1983]

Every minimal nontrivial quasigroup identity is isoparastrophic to exactly one of the following seven identities:

$x(x \cdot xy) = y$	the 1-st Belousov law [1983];
$y(x \cdot xy) = x$	the 2-nd Belousov law [1983];
$x \cdot xy = yx$	the 1-st Stein law [1957];
$xy \cdot x = y \cdot xy$	the 2-nd Stein law [1957];
$xy \cdot y = x \cdot xy$	the 1-st Shröder law [1954];
$xy \cdot yx = x$	the 2-nd Shröder law [1954];
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Theorem. [Krainichuk, 2013]

1) Shröder laws are totally symmetric; 2) the 2-nd Belousov law and the 1-st Stein law are asymmetric; 3) the 1-st Belousov law as well as the 2-nd and the 3-d Stein laws are one-side symmetric.

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Distributive quasigroups

Theorem 1.

Left distributivity has six parastrophic identities, but they are equivalent to three of them:

$$x \cdot yz = xy \cdot xz, \quad yz \cdot x = yx \cdot zx, \quad (yz)\backslash x = (y\backslash x) \cdot (z\backslash x). \quad (*)$$

Theorem 2.

Any two identities from $(*)$ imply the third one.

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The class of all distributive quasigroups are totally symmetric.

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The following theorem is some refinement of the well-known Belousov's theorem.

Theorem 3.

A quasigroup $(Q;\cdot)$ is distributive if and only if there exists a Commutative Moufang Loop $(Q;+)$ and its commuting automorphisms φ, ψ such that

$$x\cdot y=\varphi x+\psi y, \qquad x+(y+z)=(\varphi x+y)+(\psi x+z).$$

On isotopy-isomorphy problem

Pseudoisomorphy

Two quasigroups are said to be left-, right-, midldle-pseudoisomorphic, if they are isotopic and two corresponding components of the isotopism coinside.

Loops $(Q_1; \cdot)$ and $(Q_2; \circ)$ are said to be:

- *left pseudoisomorphic*, if there exists an element $c \in Q_2$ and a bijection $\theta : Q_1 \to Q_2$ such that

$$c \cdot \theta(x \cdot y) = (c \cdot \theta x) \cdot \theta y;$$

- *right pseudoisomorphic*, there exists an element $c \in Q_2$ and a bijection $\theta : Q_1 \to Q_2$ such that the equality

$$\theta(x \cdot y) \cdot c = \theta x \cdot (\theta y \cdot c).$$

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If IP-loops are left-pseudoisomorphic then they are right pseudoisomorphic and vice versa. We will say that they are peudoisomorphic.

Theorem. *1. Two isotopic IP-loops are pseudoisomorphic. 2. Two isotopic commutative IP-loops are isomorphic. 3. If a loop is isotopic to a Moufang loop then the loops are pseudoisomorphic.*

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Thank you for your attention