Free Steiner loops

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Third Mile High Conference on Nonassociative Mathematics
Denver, August 11-17, 2013

A *Steiner triple system* is an incidence structure consisting of points and blocks such that:

- every two distinct points are contained in precisely one block,
- any block has precisely three points.

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Ganter, Pfüller (1985):

The variety of all diassociative loops of exponent 2 is precisely the variety of all Steiner loops which are in a one-to-one correspondence with Steiner triple systems.

Let X be a finite ordered set, $N_a(X)$ be a set of non-associative X-words and $S(X)^* \subset N_a(X)$ be the set of S-words:

- $X \subset S(X)^*$,
- $wv \in S(X)^*$ precisely if, $v, w \in S(X)^*$, $|v| \leq |w|$, $v \neq w$ and if $w = w_1 \cdot w_2$, then $v \neq w_i$, $(i = 1, 2)$.

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$S(X) = S(X)^* \bigcup \{\emptyset\}$:

1. $v \cdot w = w \cdot v = vw$ if $vw \in S(X)$,
2. $(vw) \cdot w = w \cdot (vw) = w \cdot (wv) = (wv) \cdot w = v$,
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The set $S(X)$ with the multiplication as above is a free Steiner loop with free generators X.

Construction

Let G be a group, H be a subgroup of G and B be a set of representatives G/H with

- $B \cap H = 1$
- $b^2 = 1, b \in B$
- for any $b_1, b_2 \in B$ there exists $b_3 \in B$ such that $b_1 b_2 = b_3 h_1$, $b_2 b_1 = b_3 h_2$, $h_1, h_2 \in H$.

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Then

- $b * b = 1$,
- $(b_1 * b_2) * b_2 = b_1$.

$(B, *)$ is a free Steiner loop.

Multiplication group

Di Paola (1969):

$S(\mathfrak{S})$ is an elementary abelian 2-group of order 2^m $\Leftrightarrow$ $\mathfrak{S}$ is isomorphic to the projective space of dimension $m-1$ over the field $GF(2)$.

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Strambach, S. (2009):

Theorem

If the product of any two distinct translations of the Steiner quasigroup has an odd order then the multiplication group of the Steiner loop of order n is either the alternating group A_n or the symmetric group S_n, depending whether n is divisible by 4 or not.

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Which groups can occur in the remaining cases?

Grishkov, Rasskazova, S. (2012):

Theorem

Let $Mlt(S(X))$ be the group of the multiplications of the free Steiner loop $S(X)$. Then

1. $Mlt(S(X)) = *_{v \in D(X)^*} C_v$ *is a free product of cyclic groups of order 2;*

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1. $Mlt(S(X)) = *_{v \in D(X)^*} C_v$ *is a free product of cyclic groups of order 2;*
2. $Mlt(S(X))$ *acts on* $S(X)$ *and* $Mlt(S(X)) = \{R_v | v \in S(X)\} Inn(S(X)))$. *Moreover,* $Inn(S(X)))$ *is a free subgroup generated by* $R_v R_w R_{vw}$, $v, w \in S(X)$.

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$Aut(STS) \cong Aut(SL)$

Automorphisms

$\varphi : X \longrightarrow S(X) : (x_1, ..., x_n) \mapsto (x_1, ..., x_i \cdot v, ..., x_n)$, with $v \in S(X \setminus x_i)$ is an automorphism of $S(X)$, called an *elementary automorphism*.

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Problem 1. Which relations do exist between X-elementary automorphisms of the loop $S(X)$?

3-generated

Theorem

Let $S(X)$ be a free Steiner loop with free generators $X = \{x_1, x_2, x_3\}$. Then the group of automorphisms $Aut(S(X))$ of the loop $S(X)$ is generated by the symmetric group S_3 and by the elementary automorphism $\varphi = e_1(x_2)$.

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Any transposition of S_3 can be written as a product of elementary automorphisms

$$(ij) = e_i(x_j)e_j(x_i)e_i(x_j).$$

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$$e_1(x_2 x_3) = (13) e_1(x_2)(123) e_1(x_2)(132) e_1(x_2)(13)$$

3-generated

$$(i-1,i)(i-1,i+1)(i-1,i) = (i-1,i+1)(i-1,i)(i-1,i+1),$$

yields

$$(e_i(x_j)e_j(x_i))^3 = 1$$

$$(i-1,i)(i-1,i+1)(i-1,i) = (i-1,i+1)(i-1,i)(i-1,i+1),$$

yields

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Conjecture

The group $Aut(S(x_1,x_2,x_3))$ *is generated by three involutions* $(12),(13)$ *and* $\varphi = e_1(x_2)$, *with relations*

$$(12)(13)(12) = (13)(12)(13),$$

$$(\varphi(12))^3 = (\varphi(13))^4 = 1.$$

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$$(\varphi(12))^3 = (\varphi(13))^4 = 1.$$

Coxeter group?

Corollary

Let $S(X)$ be a free Steiner loop with free generators $X = \{a, b, c\}$ and let Q be the stabilizer $Stab_{Aut(S(X))}(c)$ of c in the automorphism group of $S(X)$. Then

$$Q =< \varphi, \tau, \xi >$$

with $\varphi(a, b, c) = (ab, b, c)$, $\xi(a, b, c) = (ac, b, c)$, $\tau(a, b, c) = (b, a, c)$.

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Theorem

If Conjecture 1 is true then Conjecture 2 is also true.

n-generated

Theorem

The automorphism group $AutD(X)$ *of the free loop* $D(X)$ *is not finite generated if* $|X| > 3$.

ISL

Let $a \in \mathfrak{S}$ be some fixed element and $IS = (\mathfrak{S}, a, \cdot)$ be a main isotope of the quasigroup associated to $\mathfrak{S}$ with multiplication

$$x \cdot y = y \cdot x = (ax)(ay).$$

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Then $x^2 = x \cdot x = (ax)(ax) = ax$ and hence $x^2 \cdot y^2 = xy$, $x^3 = x(ax) = a$, and $(xy)y = (x^2 \cdot y^2)^2 \cdot y^2 = x$.

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Conversely, from a commutative loop S with identities $x^3 = a, (x^2y^2)^2y^2 = x$ one can recover a Steiner triple system with the blocks:

- $\{x, y, x^2y^2\}$
- $\{a, x, x^2\}$

for any $x \neq y \neq a$.

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A loop obtained in this way is called an *interior Steiner loop*.

Theorem

Let $Q(X)$ be a free Steiner quasigroup with free generators X, let $S(X) = Q(X) \cup e$ and $IS(X)$ be the corresponding free exterior and interior Steiner loop, respectively.

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Then

$$Aut(Q(X)) = Aut(S(X))$$

and

$$Aut(IS(X)) \simeq Stab_{Aut(S(X))}(a)$$

where $a \in IS(X)$ is the unit element of loop $IS(X)$.

Thank you for your attention!