Novel construction of Loday-type algebras

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The varieties of dialgebras or Loday-type algebras (Leibniz, diassociative, Jordan-Loday, etc.) have been the subject of recent developments. In [KP] P. S. Kolesnikov and A.P. Pozhidaev provided a construction via conformal algebras of these varieties.

In [BFO] M. Bremner, R. Felipe and J. Sánchez-Ortega formulated a general Kolesnikov-Pozhidaev (KP) algorithm for defining the variety of $n$-ary Loday algebras (binary, triple, etc.).
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In this talk, we present a simple algorithm based on bimodules over an algebra of a given variety, and equivariant maps between the bimodule and the algebra.

This approach is equivalent to the KP algorithm for dialgebras and it allows to develop structure theory and to study properties of dialgebras.
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generalization of the Lie algebras.

Definition

A Leibniz algebra is a vector space $L$ over $K$ with a bilineal product 
called Leibniz bracket $[\cdot, \cdot] : L \times L \to L$ satisfying the Leibniz identity 
$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$, for all $x, y, z$ in $L$. 
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An associative dialgebra $D$ is a vector space over $K$ with two associative products $\triangleright$ and $\triangleleft$ satisfying for all $x, y, z$ in $D$:

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x \triangleright (y \triangleleft z) = (x \triangleright y) \triangleleft z,
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(x \triangleright y) \triangleright z = (x \triangleleft y) \triangleright z.
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The set $D$ with the bracket $[x, y] = x \triangleright y - y \triangleleft x$ is a Leibniz algebra. Moreover, J. L. Loday proved that the following diagram commutes

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\begin{array}{ccc}
\text{As} & \overset{\rightarrow}{\rightarrow} & \text{Lie} \\
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In the Jordan case, the Jordan-Loday algebras (Jordan dialgebras) satisfy the identities (see R. Velásquez and R. Felipe [VF], P. S. Kolesnikov [K1] and M. Bremner [B])

\[ x(yz) = x(zy), \quad (yx^2)x = (yx)x^2 \quad \text{and} \quad (z, y, x^2) = 2(zx, y, x) \]

Also the notions of alternative and commutative dialgebras were introduced by D. Liu in [Liu] and F. Chapoton in [C], respectively.

These notions correspond to a more general structure. P. S. Kolesnikov in [K1] and A. P. Pozhidaev in [P] provided a systematic construction for diverse varieties of dialgebras, i.e. associative, commutative, Lie (Leibniz), Jordan (restrictive quasi-Jordan), alternative, etc.
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In general, for each variety of algebras over a field, there is a definition of a corresponding variety of dialgebras (Loday-type algebras), and these varieties are constructed through the KP algorithm (see [BFS]) and BSO algorithm (see [BS]). These algorithms are generalized for \( n \)-ary Loday algebras in [BFS].
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Let $I$ be a set of multihomogeneous polynomials in $K[X]$ (the free non associative $K$-algebra generated by $X$) and let $f(x_1, ..., x_l) \in I$, with

$$
\text{deg} f = h_1 + ... + h_l \quad \text{and} \quad \text{deg}_{x_i} f = h_i \geq 1, \quad i = 1, ..., l.
$$

We define $f_{ij}(x_1, ..., x_l, y)$ as the component of $f(x_1, ..., x_i + y, ..., x_l)$ of degree $j$ in the variable $y$, for $i = 1, ..., l$ and $j = 1, ..., h_i - 1$.

1. If $\text{char} K \geq h_i$ or $\text{char} K = 0$ then $f_{ij} \in (f)$.
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From $I$, we can construct a set $I_L$ of multilinear homogenous polynomials by an iterated use of the procedure to obtain from each $f$ the $f_{ij}$'s.
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If $\text{char} K > deg_{x_i} f$, for all $f(x_1, ..., x_l) \in I$ and $i = 1, ..., l$, or $\text{char} K = 0$; we have that $(I) = (I_L)$. In these cases, the varieties of algebras $V(I)$ and $V(I_L)$ are the same.
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If $\text{char} K > \deg x_i f$, for all $f(x_1, ..., x_l) \in I$ and $i = 1, ..., l$, or $\text{char} K = 0$; we have that $(I) = (I_L)$. In these cases, the varieties of algebras $V(I)$ and $V(I_L)$ are the same.
Let $A \in \mathcal{V}(I)$ and let $M$ be an $I$-bimodule over $A$. That is, there are bilinear compositions

$$A \times M \to M; (a, m) \mapsto am \in M$$

$$M \times A \to M; (n, b) \mapsto mb \in M$$

such that $(A \oplus M, \cdot) \in \mathcal{V}(I)$, with $(a \oplus m) \cdot (b \oplus n) = ab \oplus (an + mb)$. 
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This is equivalent to $M$ satisfying the identities (see N. Jacobson [J])

$$f_i(a_1, \ldots, a_l, m); \forall f \in I; \forall i = 1, \ldots, l; \forall a_1, \ldots, a_l \in A \text{ and } \forall m \in M.$$
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$$f_{i_1}(a_1, \ldots, a_l, m); \quad \forall f \in I; \quad \forall i = 1, \ldots, l; \quad \forall a_1, \ldots, a_l \in A \quad \text{and} \quad \forall m \in M.$$

Let $\xi : M \to A; m \mapsto \xi(m)$ be an equivariant map,
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$$f_{i1}(a_1, ..., a_l, m); \ \forall f \in I; \ \forall i = 1, ..., l; \ \forall a_1, ..., a_l \in A \ \text{and} \ \forall m \in M.$$

Let $\xi : M \to A; m \mapsto \xi(m)$ be an equivariant map, that is, $\xi$ is a linear map satisfying for all $a \in A$ and $m \in M$:

$$\xi(am) = a\xi(m) \quad \text{and} \quad \xi(ma) = \xi(m)a$$
If $\xi$ is a surjective equivariant map on $M$, we define the products in $M$ by

\[
\{\cdot, \cdot\}_1 : M \times M \to M, \quad \{m, n\}_1 := m\xi(n)
\]

\[
\{\cdot, \cdot\}_2 : M \times M \to M, \quad \{m, n\}_2 := \xi(m)n
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The structure $(M; \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2)$ satisfies the identities $\hat{f}_{i1}(n_1, \ldots, n_l, m)$ obtained from $f_{i1}(\xi(n_1), \ldots, \xi(n_l), m)$ by replacing:
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3. $(\xi(t)s)$ by $\{t, s\}_2$,
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The last identities imply the 0-identities, for all $m, n, s \in M$:

$$\{m, \{n, s\}_1\}_1 = \{m, \{n, s\}_2\}_1 \quad (Id_{01})$$

$$\{\{m, n\}_1, s\}_2 = \{\{m, n\}_2, s\}_2, \quad (Id_{02})$$
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Accordingly, \((M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2)\) belongs to the variety of dialgebras \(\mathcal{V}(\hat{I})\), where 
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\hat{I} = \{\hat{f}_i(n_1, ..., n_l, m)|f(x_1, ..., x_l) \in I, i = 1, ..., l\} \cup \{Id_{01}, Id_{02}\}.
\]

Remark

We prove that the variety of dialgebras \(\mathcal{V}(\hat{I})\) so obtained from \(\mathcal{V}(I)\) is the one obtained by Kolesnikov-Pozhidaev (KP) algorithm for producing a variety of dialgebras \(\mathcal{V}(\tilde{I}_L^{KP})\) from a variety of algebras \(\mathcal{V}(I_L)\), i.e. 
\[
\mathcal{V}(\hat{I}) = \mathcal{V}(\tilde{I}_L^{KP}), \text{ if } \text{char}K > \deg_{x_i}f, \text{ for all } f(x_1, ..., x_l) \in I \text{ and } i = 1, ..., l, \text{ or } \text{char}K = 0.
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*We prove that the variety of dialgebras \(\mathcal{V}(\hat{I})\) so obtained from \(\mathcal{V}(I)\) is the one obtained by Kolesnikov-Pozhidaev (KP) algorithm for producing a variety of dialgebras \(\mathcal{V}(\tilde{I}_L^{KP})\) from a variety of algebras \(\mathcal{V}(I_L)\), i.e. \(\mathcal{V}(\hat{I}) = \mathcal{V}(\tilde{I}_L^{KP})\), if \(\text{char} K > \deg_{x_i} f\), for all \(f(x_1, \ldots, x_l) \in I\) and \(i = 1, \ldots, l\), or \(\text{char} K = 0\).*

The present formalism allows to study the classification of Loday algebras through the representations of algebras.
Accordingly, \((M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2)\) belongs to the variety of dialgebras \(\mathcal{V}(\hat{I})\), where 
\[
\hat{I} = \{ \hat{f}_i(n_1, \ldots, n_l, m) \mid f(x_1, \ldots, x_l) \in I, i = 1, \ldots, l \} \cup \{ \text{Id}_{01}, \text{Id}_{02} \}.
\]

**Remark**

We prove that the variety of dialgebras \(\mathcal{V}(\hat{I})\) so obtained from \(\mathcal{V}(I)\) is the one obtained by Kolesnikov-Pozhidaev (KP) algorithm for producing a variety of dialgebras \(\mathcal{V}(	ilde{I}_L^{KP})\) from a variety of algebras \(\mathcal{V}(I_L)\), i.e. 
\[
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\]

The present formalism allows to study the classification of Loday algebras through the representations of algebras.
Let’s consider the set of identities $I = \{xy = yx, (x^2y)x = x^2(yx)\}$. An algebra $A$ is an element of $\mathcal{V}(I)$ if for all $a, b \in A$, we have $ab =.ba$ and $(a^2b)a = a^2(ba)$. The variety $\mathcal{V}(I)$ is the variety of Jordan algebras.

If $M$ is an $I$-bimodule over $A$ then from the equation $xy = yx$ we get

$$am = ma \quad (1)$$
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(ma^2)a = (ma)a^2
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(2)

and the associator (Osborn) identity

\[
2(ma, b, a) = (m, b, a^2)
\]

(3)
Let’s consider the set of identities $I = \{xy = yx, (x^2y)x = x^2(yx)\}$. An algebra $A$ is an element of $\mathcal{V}(I)$ if for all $a, b \in A$, we have $ab = ba$ and $(a^2b)a = a^2(ba)$. The variety $\mathcal{V}(I)$ is the variety of Jordan algebras.

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If $\xi : M \to A$ is an equivariant surjective map, from (1) we have that
\[
\{n, m\}_1 = \{m, n\}_2 := nm
\]
Because of the definition of the equivariant map
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n\xi(m\xi(v)) = n(\xi(m)\xi(v)) = n(\xi(v)\xi(m)) = n\xi(v\xi(m))
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Hence, $(M; \cdot) \in \mathcal{V}(\hat{I})$, where $\hat{I} = \{J0, J1, J2\}$, this is
\[
\hat{I} = \{x(yz) - x(zy), (xy^2)y - (xy)y^2, 2(xy, z, y) = (x, z, y^2)\}.
\]
If $\xi : M \rightarrow A$ is an equivariant surjective map, from (1) we have that
\[
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\]
and so, we have the 0-identity
\[
n(m\nu) = n(\nu m) \quad (J0)
\]
Now, from (2), we have the Jordan identity
\[
(mn^2)n = (mn)n^2 \quad (J1)
\]
Finally, from (3), we obtain the Osborn identity
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Hence, $(M; \cdot) \in \mathcal{V}(\hat{I})$, where $\hat{I} = \{J0, J1, J2\}$, this is
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There remain to prove if for every dialgebra \((M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2) \in \mathcal{V}(\tilde{I}_L^{KP})\), we have that:

1. \(M\) is an \(I\)-bimodule over an algebra \(A \in \mathcal{V}(I_L)\).
2. There exists a surjective equivariant map \(\xi\) such that \(\{m, n\}_1 = m\xi(n)\) and \(\{m, n\}_2 = \xi(m)n\).
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For \((M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2) \in \mathcal{V}(\widetilde{I}^{KP}_L)\), we define

\[
\text{Ann}(M) := \langle \{m, n\}_1 - \{m, n\}_2 | m, n \in M \rangle
\]

and

\[
Z_B(M) := \left\{ m | \{n, m\}_1 = 0 \text{ and } \{m, s\}_2 = 0, \forall n, s \in M \right\}
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We have that $\text{Ann}(M)$ and $\text{Z}_B(M)$ are ideals of $M$, $\text{Ann}(M) \subseteq \text{Z}_B(M)$ and

$$\{\text{Z}_B(M), M\}_1 + \{M, \text{Z}_B(M)\}_2 \subseteq \text{Ann}(M).$$

**Theorem**

Let $(M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2) \in \mathcal{V}(I_L^{KP})$ be a dialgebra. Then we have that the quotient algebras

$$\overline{M} := M/\text{Ann}(M) \quad \text{and} \quad \widehat{M} := M/\text{Z}_B(M)$$

are in the variety $\mathcal{V}(I_L)$ and $M$ is a $I_L$-bimodule over $\overline{M}$ and over $\widehat{M}$ respectively.
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The canonical maps $\overline{\pi}$ and $\hat{\pi}$ are surjective equivariant maps of $M$ as $I_L$-bimodule over $\overline{M}$ and $\hat{M}$, respectively.
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The canonical maps $\overline{\pi}$ and $\hat{\pi}$ are surjective equivariant maps of $M$ as $I_L$-bimodule over $\overline{M}$ and $\hat{M}$, respectively.
Theorem

Let $M$ be an $I$-bimodule over $A$ and let $\xi : M \to A$ a surjective equivariant map. For the dialgebra $(M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2)$ we have that

$\bullet$ The map $\phi : \overline{M} \to A$, defined by $\overline{m} \mapsto \xi(m)$ is a surjective algebra homomorphism.

Moreover, $\phi$ is an isomorphism if the condition $Am \subseteq \text{Ann}(M)$ or $mA \subseteq \text{Ann}(M)$ implies that $m \in \text{Ann}(M)$, is satisfied for all $m \in M$. 
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2. The map $\theta : A \to \hat{M}$, defined by $\xi(m) \mapsto \hat{m}$ is a surjective algebra homomorphism.
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3. The set
   $$I := \xi^{-1}(\{0\}) = \left\{ m \in M | \xi(m) = 0 \right\}$$
   is an ideal of $M$ such that $\text{Ann}(M) \subseteq I \subseteq \mathbb{Z}_B(M)$ and the quotient algebra $\tilde{M} := M/I$ is isomorphic to $A$. 
The homomorphism theorems, bar units and derivations

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For a dialgebra \((M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2) \in \mathcal{V}(\tilde{I}_L^{KP})\) we say that \(e \in M\) is a **bar unit** if \(\{m, e\}_1 = m\) and \(\{e, n\}_2 = n\), for all \(m, n \in M\); and we denote \(H(M)\) the set of bar units in \(M\) (**the halo of** \(M\)).

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**Theorem**

Let \(M\) be an \(I\)-bimodule over \(A\) and let \(\xi : M \to A\) a surjective equivariant map. For the dialgebra \((M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2)\) we have that

\[ \text{The algebra } A \text{ is unital iff } H(M) \neq \Phi. \text{ In this case, } H(M) = \text{Ker}_1\xi = \{m \in M | \xi(m) = 1 \in A\}. \]
The homomorphism theorems, bar units and derivations

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1. The algebra \(A\) is unital iff \(H(M) \neq \emptyset\). In this case, \(H(M) = \text{Ker}_1\xi = \{m \in M \mid \xi(m) = 1 \in A\}\).

2. If \(H(M) \neq \emptyset\), then \(\text{Ann}(M) = Z_B(M)\) and both \(\phi\) and \(\theta\) are isomorphisms.
For a dialgebra \((M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2) \in \mathcal{V}(\tilde{I}^{KP}_L)\) we say that \(e \in M\) is a bar unit if \(\{m, e\}_1 = m\) and \(\{e, n\}_2 = n\), for all \(m, n \in M\); and we denote \(H(M)\) the set of bar units in \(M\) (the halo of \(M\)).

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3. If \(e \in H(M)\), then \(H(M) = \{e + m | m \in \text{Ann}(M)\}\).
For a dialgebra \((M, \{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2) \in \mathcal{V}(\tilde{I}_L^{KP})\) we say that \(e \in M\) is a **bar unit** if \(\{m, e\}_1 = m\) and \(\{e, n\}_2 = n\), for all \(m, n \in M\); and we denote \(H(M)\) the set of bar units in \(M\) (**the halo of** \(M\)).

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On derivations over dialgebras

Let $M$ an $I$-bimodule over $A$, $\xi : M \rightarrow A$ a surjective equivariant map and $D : A \rightarrow M$ a derivation, i.e a linear map such that

$$D(ab) = D(a) \cdot b + a \cdot D(b), \quad \forall a, b \in A.$$
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Then for any $a, b \in A$ and $m, n \in M$ we have that

$$m \cdot a \xrightarrow{\xi} \xi(m)a \xrightarrow{D} D(\xi(m)) \cdot a + \xi(m) \cdot D(a)$$

and

$$b \cdot n \xrightarrow{\xi} b\xi(n) \xrightarrow{D} D(b) \cdot \xi(n) + b \cdot D(\xi(n)).$$
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Then for any $a, b \in A$ and $m, n \in M$ we have that

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Hence, the definition of products $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ implies that

$$\{m, n\}_1 \xmapsto{} \{D(\xi(m)), n\}_1 + \{m, D(\xi(n))\}_2$$

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Hence, the definition of products $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ implies that

$$\{m, n\}_1 \overset{D}{\mapsto} \{D(\xi(m)), n\}_1 + \{m, D(\xi(n))\}_2$$

and

$$\{m, n\}_2 \overset{D}{\mapsto} \{D(\xi(m)), n\}_1 + \{m, D(\xi(n))\}_2,$$

for any $m, n \in M$. 
Then we have defined a linear map $\delta : M \to M$ such that

$$\delta(\{m, n\}; = \{\delta(m), n\}_1 + \{m, \delta(n)\},$$

for $i = 1, 2$ and for all $m, n \in M$.

We call this maps a diderivations and we denote by $Dider(M)$ the set of diderivations over $M$.

1. If $A$ is a Lie algebra then $\{m, n\}_1 = -\{n, m\}_2$ and we have the notion of anti-derivation over Leibniz algebras (introduced by J. L. Loday).
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We call this maps a diderivation and we denote by $Dider(M)$ the set of diderivations over $M$.

1. If $A$ is a Lie algebra then $\{m, n\}_1 = -\{n, m\}_2$ and we have the notion of anti-derivation over Leibniz algebras (introduced by J. L. Loday).

2. If $A$ is a Jordan algebra then $\{m, n\}_1 = \{n, m\}_2$ and we have the notion of left-derivation respect to the product $\{m, n\}_1$ and the right-derivation respect to the product $\{m, n\}_2$ over Jordan-Loday algebras introduced by (R. Velásquez and R. Felipe). Moreover, $[L_m, R_m]$ is a inner left-derivation (rigth-derivation) respect to the correspondent product.
Then we have defined a linear map $\delta : M \to M$ such that

$$\delta(\{m, n\}; = \{\delta(m), n\}_1 + \{m, \delta(n)\},$$

for $i = 1, 2$ and for all $m, n \in M$. We call this maps a diderivations and we denote by $Dider(M)$ the set of diderivations over $M$.

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3 If $A$ is an associative algebra, R. Velásquez and G. Restrepo defined the definition of diderivations by dialgebras and showed that $L^i_m - R^i_m$ are inner derivations and $L^1_m - R^2_m$ is a diderivation over any dialgebra associative.

4 Finally, in the three cases we have that $Dider(M)$ is a Lie module over $Der(M)$, for any product, with respecto to the bracket

$$[\delta, D] := \delta D - D\delta$$
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