Novel constructionof Loday-type algebras

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Definition

A **Leibniz algebra** is a vector space L over K with a bilineal product called **Leibniz bracket** $[\cdot,\cdot] : L \times L \to L$ satisfying the *Leibniz identity* $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$, for all x, y, z in L.

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An **associative dialgebra** D is a vector space over K with two associative products $\vdash$ and $\dashv$ satisfying for all x, y, z in D:

$x \dashv (y \dashv z) = (x \dashv y) \dashv z,$

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The set D with the bracket $[x, y] = x \vdash y - y \dashv x$ is a Leibniz algebra. Moreover, J. L. Loday proved that the following diagram commutes

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In the Jordan case, the Jordan-Loday algebras (Jordan dialgebras) satisfy the identities (see R. Velásquez and R. Felipe [*VF*], P. S. Kolesnikov [*K*1] and M. Bremner [*B*])

$$x(yz) = x(zy), \quad (yx^2)x = (yx)x^2 \ \text{ and } \ (z, y, x^2) = 2(zx, y, x)$$

Also the notions of alternative and commutative dialgebras were introduced by D. Liu in [*Liu*] and F. Chapoton in [*C*], respectively.

These notions correspond to a more general structure. P. S. Kolesnikov in [*K*1] and A. P. Pozhidaev in [*P*] provided a systematic construction for diverse varieties of dialgebras, i.e. associative, commutative, Lie (Leibniz), Jordan (restrictive quasi-Jordan), alternative, etc.

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In general, for each variety of algebras over a field, there is a definition of a corresponding variety of dialgebras (Loday-type algebras), and these varieties are constructed through the KP algorithm (see [*BFS*]) and BSO algorithm (see [*BS*]). These algorithms are generalized for n-ary Loday algebras in [*BFS*].

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Let I be a set of multihomogeneus polynomials in $K[X]$ (the free non associative K-algebra generated by X) and let $f(x_1, ..., x_l) \in I$, with

$$degf = h_1 + ... + h_l \quad \text{and} \quad deg_{x_i} f = h_i \geq 1, \quad i = 1, ..., l.$$

We define $f_{ij}(x_1, ..., x_l, y)$ as the component of $f(x_1, ..., x_i + y, ..., x_l)$ of degree j in the variable y, for $i = 1, ..., l$ and $j = 1, ..., h_{i-1}$.

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Let $A \in \mathcal{V}(I)$ and let M be an I-bimodule over A. That is, there are bilinear compositions

$$A \times M \to M; (a, m) \mapsto am \in M$$

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If ξ is a **surjective** equivariant map on M, we define the products in M by

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The last identities imply the 0-identities, for all $m, n, s \in M$:

$$\{m, \{n,s\}_1\}_1 = \{m, \{n,s\}_2\}_1 \qquad (Id_{01})$$

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Accordingly, $(M, \{\cdot,\cdot\}_1, \{\cdot,\cdot\}_2)$ belongs to the variety of dialgebras $\mathcal{V}(\hat{I})$, where $\hat{I} = \{\hat{f}_{i1}(n_1, ..., n_l, m) | f(x_1, ..., x_l) \in I, i = 1, ..., l\} \cup \{Id_{01}, Id_{02}\}$.

Remark

We prove that the variety of dialgebras $\mathcal{V}(\hat{I})$ so obtained from $\mathcal{V}(I)$ is the one obtained by Kolesnikov-Pozhidaev (KP) algorithm for producing a variety of dialgebras $\mathcal{V}(\widetilde{I}_L^{KP})$ from a variety of algebras $\mathcal{V}(I_L)$, i.e. $\mathcal{V}(\hat{I}) = \mathcal{V}(\widetilde{I}_L^{KP})$, if $charK > deg_{x_i} f$, for all $f(x_1, ..., x_l) \in I$ and $i = 1, ..., l$, or $charK = 0$.

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and from $(x^2y)x = x^2(yx)$, using (1), we obtain

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Let's consider the set of identities $I = \{xy = yx, (x^2y)x = x^2(yx)\}$.
An algebra A is an element of $\mathcal{V}(I)$ if for all $a, b \in A$, we have $ab = ba$ and $(a^2b)a = a^2(ba)$. The variety $\mathcal{V}(I)$ is the variety of Jordan algebras.

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If $\xi : M \to A$ is an equivariant surjective map, from (1) we have that $\{n, m\}_1 = \{m, n\}_2 := nm$

Because of the definition of the equivariant map

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There remain to prove if for every dialgebra $(M, \{\cdot,\cdot\}_1, \{\cdot,\cdot\}_2) \in \mathcal{V}(\widetilde{I}_L^{KP})$, we have that:

1. M is an I-bimodule over an algebra $A \in \mathcal{V}(I_L)$.
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$$Ann(M) := \langle \{m,n\}_1 - \{m,n\}_2 | m,n \in M \rangle$$

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We have that $Ann(M)$ and $Z_B(M)$ are ideals of M, $Ann(M) \subseteq Z_B(M)$ and

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Theorem

Let $(M, \{\cdot,\cdot\}_1, \{\cdot,\cdot\}_2) \in \mathcal{V}(\widetilde{I}_L^{KP})$ be a dialgebra. Then we have that the quotient algebras

$$\overline{M} := M/Ann(M) \quad and \quad \widehat{M} := M/Z_B(M)$$

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For a dialgebra $(M, \{\cdot,\cdot\}_1, \{\cdot,\cdot\}_2) \in \mathcal{V}(\widetilde{I}_L^{KP})$ we say that $e \in M$ is a **bar unit** if $\{m, e\}_1 = m$ and $\{e, n\}_2 = n$, for all $m, n \in M$; and we denote $H(M)$ the set of bar units in M (**the halo of** M).

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On derivations over dialgebras

Let M an I-bimodule over A, $\xi : M \to A$ a surjective equivarient map and $D : A \to M$ a derivation, i.e a linear map such that

$$D(ab) = D(a) \cdot b + a \cdot D(b), \quad \forall a, b \in A.$$

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Then we have defined a linear map $\delta : M \to M$ such that

$$\delta(\{m,n\}_i = \{\delta(m), n\}_1 + \{m, \delta(n)\},$$

for $i = 1, 2$ and for all $m, n \in M$.
We call this maps a diderivations and we denote by $Dider(M)$ the set of diderivations over M.

1. If A is a Lie algebra then $\{m,n\}_1 = -\{n,m\}_2$ and we have the notion of anti-derivation over Leibniz algebras (introduced by J. L. Loday).

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