A simple approach to n-ary Loday algebras

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Third Mile High Conference on Nonassociative Mathematics 2013

Denver, Colorado, August 12 - 16

August 12, 2013

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Let $f_{ij}(x_1, ..., x_l, y)$ be a component of $f(x_1, ..., x_{i-1}, x_i + y, x_{i+1}, ..., x_l)$ of degree j in the variable y, for each $i = 1, ..., l$ and $j = 1, ..., h_i$, then

1. If $charK \geq h_i$ or $charK = 0$ then $f_{ij} \in (f); j = 1, ..., h_i$.
2. If $charK > \binom{h_i}{j}$ or $charK = 0$ then $f \in (f_{ij}); j = 1, ..., h_i$.

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From the set I of multihomogeneus polynomials we can construct a set I_L of multilinear homogenous polynomials by an iterated use of the procedure to obtain from each f the $f_{ij}'s$.

If $charK > deg_{x_i} f$ or $charK = 0$ then $(f) = (f_{ij})$, $j = 1, ..., h_i$, and we see that the ideal generated by the **set I_L of linearized equations**, obtained from the iterated use of $f \mapsto f_{ij}$, coincides with the ideal generated by I.

Hence, under the constraint $(charK > deg_{x_i} f, \forall f(x_1, ..., x_l) \in I, i = 1, ..., l)$ or $charK = 0$ the construction starting from (I) or from (I_L) are equivalent.

Let $(A, (\cdot, \cdots, \cdot)_a)$ be an n**-ary algebra over** K. The algebra belongs to the **variety generated by the set of multihomogeneous** n**-ary identities** I, i.e., $A \in \mathcal{V}(I)$.

A K-space M is an n**–ary** I**-module over** A if there are n multilinear compositions $P_{at} : A^{t-1} \times M \times A^{n-t} \to M$ that send

$$(a_1, ..., a_{t-1}, m, a_{t+1}, ..., a_n) \mapsto (a_1, ..., a_{t-1}, m, a_{t+1}, ..., a_n)_{a_t}, \ \forall t = 1, ..., n,$$

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such that $A \oplus M \in \mathcal{V}(I)$, i.e. for each $f(x_1, ..., x_l) \in I$ we have

$$\widetilde{f}_{i1}(a_1, ..., a_n, m) = 0 \ \ \forall a_1, ..., a_n \in A \ \text{ and } \ \forall m \in M,$$

where $\widetilde{f}_{i1}(x_1, ..., x_n, y)$ is the identity obtained from $f_{i1}(x_1, ..., x_n, y)$, by replacing $(v_1, ..., v_n)_a$ by $(v_1, ..., v_n)_{a_t}$ whenever y appears in the term v_t and by $(v_1, ..., v_n)_a$ whenever y does not appear in any of the terms $v_1, ..., v_n$.

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We call the linear map $\xi : M \to A$ an **equivariant map** whenever

$$\xi((a_1, ..., a_{t-1}, m, a_{t+1}, ..., a_n)_{a_t}) = (a_1, ..., a_{t-1}, \xi(m), a_{t+1}, ..., a_n)_a,$$

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Let M be an n–ary I-module over A, with multilinear compositions $P_{at}; t = 1, ..., n$, and let $\xi : M \to A$ be a **surjective** equivariant map.

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$$\{m_1, ..., m_n\}_{a_t} := (\xi(m_1), ..., \xi(m_{t-1}), m_t, \xi(m_{t+1}), ..., \xi(m_n))_{a_t}.$$

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Hence, the products in M satisfy the following set of identities for $i = 1, ..., n,$:

$$\{m_1, \cdots, m_{j-1}, \{ s_1, ..., s_n\}_{a_t}, m_{j+1}, ..., m_n\}_{a_i} \\ = \{m_1, \cdots, m_{j-1}, \{ s_1, ..., s_n\}_{a_r}, m_{j+1}, , m_n\}_{a_i}, \qquad (Id_{0i})$$

where $j \neq i$, for all $j, t, r \in \{1, 2, \ldots, n\}$.

That is, a product $\{s_1, ..., s_n\}_{a_t}$ as an entry in a position different from the i-th in $\{\cdot, ..., \cdot\}_{a_i}$ can be exchanged by $\{s_1, ..., s_n\}_{a_r}$ for any $t, r = 1, ..., n$.

An iterated use of this property leads to the conclusion that any products involved in an entry different from the i-th in a product of the form $\{\cdot, ..., \cdot\}_{a_i}$can be changed by any other type of products as long as their arguments are kept fixed. The identities $(\mathrm{Id}_{01}), ..., (\mathrm{Id}_{0n})$ are called the set of 0-th identities.

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The algebra M with the n different n-ary products $(M; \{\cdot, ..., \cdot\}_{a_1}, ..., \{\cdot, ..., \cdot\}_{a_n})$ fulfills, besides the 0-th identities, for each $f \in I$, the identities $\hat{f}_{i_1}(x_1, ..., x_l, y)$ obtained from $\widetilde{f}_{i1}(x_1, ..., x_l, y)$ by replacing

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2. $(\cdot, ..., \cdot)_a$ by $\{\cdot, ..., \cdot\}_{a_1}$ (it can be replaced by any $\{\cdot, ..., \cdot\}_{a_r}$ since the 0-th identities hold).

Hence, $(M; \{\cdot, ..., \cdot\}_{a_1}, ..., \{\cdot, ..., \cdot\}_{a_n})$ belongs to the **variety of n-ary Loday algebras** $\mathcal{V}(\hat{I})$ where

$$\hat{I} = \{(Id_{01}), \cdots, (Id_{0n})\} \cup \{\hat{f}_{i1}(x_1, ..., x_l, y) | f(x_1, ...x_l) \in I, i = 1, ..., l\}$$

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By an argument analogous to the binary algebras case we verify that M belongs to the generalized variety of n-ary Loday algebras $\mathcal{V}(\tilde{I}_L^{KP})$ obtained by the generalized KP algorithm whenever

$$charK > deg_{x_i} f, \quad \text{for all} \quad f \in I, \quad \text{or} \quad charK = 0,$$

where I_L is a multilinear set of identities such that $(I_L) = (I)$.

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According to the KP algorithm, given a multilinear polynomial identity $f(x_1, ..., x_l) \in I_L$, we obtain an identity $\widetilde{f}_i$ in $\widetilde{I}_L^{KP}$ for each $i = 1, ..., l$ through the following procedure:

1. For each adopted $i \in \{1, ..., l\}$ we substitute the n–ary products in $f(m_1, ..., m_l)$; $m_1, ..., m_l \in M$ with the n–ary products in M, $\{\cdot, \cdots, \cdot\}_{a_1}, \cdots, \{\cdot, \cdots, \cdot\}_{a_n}$ to obtain $\widetilde{f}_i$ using the rules:

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Observe that

$$\hat{f}_{i1}(m_1, ..., m_{i-1}, 0, m_{i+1}, ..., m_l, m_i)$$

is obtained from

$$f_{i1}(\xi(m_1), ..., \xi(m_{i-1}), 0, \xi(m_{i+1}), ..., \xi(m_l), m_i),$$

after we perform the substitutions:

- $\{\xi(\eta_1), ..., \xi(\eta_{j-1}), t, \xi(\eta_{j+1}), ..., \xi(\eta_n)\}_a$ by $\{\eta_1, ..., \eta_{j-1}, t, \eta_{j+1}, ..., \eta_n\}_{a_t}$ when m_i occurs within the factor t.

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- $\{\xi(\eta_1), \cdots, \xi(\eta_n)\}_a$ by $\xi(\{\eta_1, \cdots, \eta_n\}_{a_1})$, when m_i does not occur within any of the η_i's factors. Actually, we can substitute it by any $\xi(\{\eta_1, \cdots, \eta_n\}_{a_j})$ due to the properties of the equivariant function. We adopt substituting by $\xi(\{\eta_1, \cdots, \eta_n\}_{a_1})$ when m_i occurs to the left and by $\xi(\{\eta_1, \cdots, \eta_n\}_{a_n})$ when m_i occurs to the right of the considered product.

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We observe that under these substitution we get

$$\widetilde{f}_i(m_1, ..., m_l) = \hat{f}_{i1}(m_1, ..., m_{i-1}, 0, m_{i+1}, ..., m_l, m_i)$$

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- $\{\xi(\eta_1), \cdots, \xi(\eta_n)\}_a$ by $\xi(\{\eta_1, \cdots, \eta_n\}_{a_1})$, when m_i does not occur within any of the η_i's factors. Actually, we can substitute it by any $\xi(\{\eta_1, \cdots, \eta_n\}_{a_j})$ due to the properties of the equivariant function. We adopt substituting by $\xi(\{\eta_1, \cdots, \eta_n\}_{a_1})$ when m_i occurs to the left and by $\xi(\{\eta_1, \cdots, \eta_n\}_{a_n})$ when m_i occurs to the right of the considered product.

We observe that under these substitution we get

$$\widetilde{f}_i(m_1, ..., m_l) = \hat{f}_{i1}(m_1, ..., m_{i-1}, 0, m_{i+1}, ..., m_l, m_i)$$

The 0-th identities assert that the label of a product within an entry different from the i-th of a product $\{\cdot, \cdots, \cdot\}_{a_i}$ can be changed arbitrarily. Hence, only when m_j is within one of the factors, the prescription is tight in regards to which product to use, and such prescription coincides in both schemes. So,

$$\widetilde{f}_i(m_1, ..., m_l) = \hat{f}_{i1}(m_1, ..., m_{i-1}, 0, m_{i+1}, ..., m_l, m_i)$$

Hence, since

$$\hat{I}_L = \{\hat{f}_{i1}(x_1, ..., x_{i-1}, 0, x_{i+1}, x_l, y) | f \in I_L, i = 1, ..., l\} \cup \{(Id_{01}), \cdots, (Id_{0n})\}$$

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Lemma

Let $charK > deg_{x_i} f$, for all $f(x_1, ..., x_l) \in I$ and $i = 1, ..., l$, or $charK = 0$. Then the varieties defined through the I_L–bimodule and the equivariant map are identical, that is $\mathcal{V}(\widetilde{I}_L^{KP}) = \mathcal{V}(\hat{I}_L)$.

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There remain to verify that how to associate an n-ary algebra in $\mathcal{V}(I)$ to each n-ary Loday algebra $M \in \mathcal{V}(\widetilde{I}_L^{KP})$ so that M is an n–ary I–module over that algebra, and associate to it an equivariant function.
More concretely, if we defined

$$Ann(M) := \langle \{ \{\eta_1, ..., \eta_n\}_{a_i} - \{\eta_1, ..., \eta_n\}_{a_j} | \eta_1, ..., \eta_n \in M; i, j = 1, ..., n\} \rangle$$

and

$$\begin{aligned} Z_B(M) := \{ t \in M | \{ \{\eta_1, ..., \eta_s, t, \eta_{s+1}, ..., \eta_n\}_{a_i} = 0, \\ \eta_j \in M, \ i, s, j \in \{1, ..., n\}, s \neq i \}, \end{aligned}$$

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Theorem

Let $M \in \mathcal{V}(\widetilde{I}_L^{KP})$ be an n-ary Loday algebra. Then $Ann(M)$ and $Z_B(M)$ are ideals in M such that $Ann(M) \subset Z_B(M)$, M is an n–ary I_L-module over $\bar{M} := M/Ann(M)$ and over $\hat{M} := M/Z_B(M)$ and the canonical maps $M \to \bar{M}$ and $M \to \hat{M}$ are equivariant maps.

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Lemma

Let $M \in \mathcal{V}(\hat{I}_L^{KP})$ be an n-ary Loday algebra. If the products in M were constructed from M as an n–ary I_L-module over the algebra $A \in \mathcal{V}(I_L)$ with the equivariant map $\xi : M \to A$, then $J_\xi := \xi^{-1}(\{0\})$ is an ideal of M such that $Ann(M) \subset J_\xi \subset Z_B(M)$ and $M/J_\xi \cong A$ as a n-ary algebra.

Example: Jordan triple system

Consider $T \in \mathcal{V}(I)$

$$\{x, y, z\} = \{z, y, x\} \qquad (TJ1)$$

$$\{x, y\{z, u, v\}\} = \{\{x, y, z\}u, v\} - \{z, \{y, x, u\}v\} + \{z, u, \{x, y, v\}\} \qquad (TJ2)$$

To describe ternary I-modules over T, consider the multilinear compositions

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To describe ternary I-modules over T, consider the multilinear compositions

$$\begin{array}{ll} M \times T \times T \to M & (m,a,b) \mapsto (m,a,b)_1 \\ T \times M \times T \to M & (a,m,b) \mapsto (a,m,b)_2 \\ T \times T \times M \to M & (a,b,m) \mapsto (a,b,m)_3 \end{array}$$

such that $T \oplus M \in \mathcal{V}(I)$ with product
$\{a \oplus m, b \oplus n, c \oplus s\} = \{a,b,c\} \oplus ((a,b,s)_1 + (a,n,c)_2 + (m,b,c)_3)$.
Identity $(TJ1)$ implies

$$(a,b,s)_3 = (s,b,a)_1 \qquad (TJL1a)$$

$$(a,m,c)_2 = (c,m,a)_2 \qquad (TJL1b)$$

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Hence, there are only two independent multilinear compositions $(\cdot,\cdot,\cdot)_1$ and $(\cdot,\cdot,\cdot)_2$, and the later only is symmetric w.r.t. the exchange of the first and last arguments.

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\begin{array}{ll}
M \times T \times T \to M & (m,a,b) \mapsto (m,a,b)_1 \\
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Hence, there are only two independent multilinear compositions $(\cdot,\cdot,\cdot)_1$ and $(\cdot,\cdot,\cdot)_2$, and the later only is symmetric w.r.t. the exchange of the first and last arguments.

Identity $(TJ2)$ implies

$$
\begin{aligned}
(m, a, \{b, c, d\})_1 = &((m, a, b)_1, c, d)_1 - (b, (a, m, c)_2, d)_2 \\
&+ ((m, a, d)_1, c, b)_1, & (TJL2a) \\
(a, m, \{b, c, d\})_2 = &((a, m, b)_2, c, d)_1 - (b, (m, a, c)_1, d)_2 \\
&+ ((a, m, d)_2, c, b)_1, & (TJL2b) \\
((m, c, d)_1, b, a)_1 = &((m, b, a)_1, c, d)_1 - (m, \{c, a, b\}, d)_1 \\
&+ (m, c, \{d, b, a\})_1, & (TJL2c) \\
((c, m, d)_2, b, a)_1 = &(\{c, b, a\}, m, d)_2 - (c, (m, a, b)_1, d)_2 \\
&+ (c, m, \{d, b, a\})_2, & (TJL2d) \\
((m, d, c)_1, b, a)_1 = &((m, b, a)_1, d, c)_1 - (m, \{d, a, b\}, c)_1 \\
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Identity $(TJ2)$ implies

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(m,a,\{b,c,d\})_1 =& ((m,a,b)_1,c,d)_1 - (b,(a,m,c)_2,d)_2 \\
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The last identity is not independent, it is $TJL2c$ after the exchange $c \leftrightarrow d$. So, $\tilde{I} := \{(TJL1a),(TJL2a),\cdots,(TJL2d)\}$, and $M \in \mathcal{V}(\tilde{I})$ the variety of ternary $\tilde{I}$–modules over A.

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& + ((m, a, d)_1, c, b)_1, && (TJL2a) \\
(a, m, \{b, c, d\})_2 =& ((a, m, b)_2, c, d)_1 - (b, (m, a, c)_1, d)_2 \\
& + ((a, m, d)_2, c, b)_1, && (TJL2b) \\
((m, c, d)_1, b, a)_1 =& ((m, b, a)_1, c, d)_1 - (m, \{c, a, b\}, d)_1 \\
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((c, m, d)_2, b, a)_1 =& (\{c, b, a\}, m, d)_2 - (c, (m, a, b)_1, d)_2 \\
& + (c, m, \{d, b, a\})_2, && (TJL2d) \\
((m, d, c)_1, b, a)_1 =& ((m, b, a)_1, d, c)_1 - (m, \{d, a, b\}, c)_1 \\
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The last identity is not independent, it is $TJL2c$ after the exchange $c \leftrightarrow d$. So, $\tilde{I} := \{(TJL1a), (TJL2a), \cdots, (TJL2d)\}$, and $M \in \mathcal{V}(\tilde{I})$ the variety of ternary $\tilde{I}$–modules over A.

Consider a surjective equivariant map $\xi : M \to T$, such that:
$\xi((m,a,b)_1) = \{\xi(m), a, b\}$,
$\xi((a,m,b)_2) = \{a, \xi(m), b\}$.

We define the products in M:
$\{m,n,r\}_1 := (m, \xi(n), \xi(r))_1$,
$\{m,n,r\}_2 := (\xi(m), n, \xi(r))_2$,
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Now, from
$\xi((m,\xi(n),\xi(r))_1) = \xi((\xi(m),n,\xi(r))_2) = \xi((\xi(m),\xi(n),r)_3) = \{\xi(m),\xi(n),\xi(r)\}$,

it follows
$\xi(\{m,n,r\}_1) = \xi(\{m,n,r\}_2) = \xi(\{r,n,m\}_1)$.

From this it follows the 0-th identities:

$$\begin{aligned} \{m,\{n,r,s\}_1,t\}_1 &= \{m,\{n,r,s\}_2,t\}_1 \\ &= \{m,\{s,r,n\}_1,t\}_1, \qquad (Id_{01}) \\ \{m,n,\{r,s,t\}_1\}_1 &= \{m,n,\{r,s,t\}_2\}_1 \\ &= \{m,n,\{t,s,r\}_1\}_1, \qquad (Id_{02}) \\ \{\{m,n,r\}_1,s,t\}_2 &= \{\{m,n,r\}_2,s,t\}_2 \\ &= \{\{r,n,m\}_1,s,t\}_2. \qquad (Id_{03}) \end{aligned}$$

Now, from
$\xi((m,\xi(n),\xi(r))_1)=\xi((\xi(m),n,\xi(r))_2)=\xi((\xi(m),\xi(n),r)_3)=\{\xi(m),\xi(n),\xi(r)\}$,

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\{m,\{n,r,s\}_1,t\}_1 &= \{m,\{n,r,s\}_2,t\}_1 \\
&= \{m,\{s,r,n\}_1,t\}_1, && (Id_{01}) \\
\{m,n,\{r,s,t\}_1\}_1 &= \{m,n,\{r,s,t\}_2\}_1 \\
&= \{m,n,\{t,s,r\}_1\}_1, && (Id_{02}) \\
\{\{m,n,r\}_1,s,t\}_2 &= \{\{m,n,r\}_2,s,t\}_2 \\
&= \{\{r,n,m\}_1,s,t\}_2. && (Id_{03})
\end{aligned}$$

Now, from the identity $TJM1b$:
$\{m,n,r\}_2=\{r,n,m\}_2. \qquad (TJL1b)$

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From this it follows the 0-th identities:

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\{m,\{n,r,s\}_1,t\}_1 &= \{m,\{n,r,s\}_2,t\}_1 \\
&= \{m,\{s,r,n\}_1,t\}_1, && (Id_{01}) \\
\{m,n,\{r,s,t\}_1\}_1 &= \{m,n,\{r,s,t\}_2\}_1 \\
&= \{m,n,\{t,s,r\}_1\}_1, && (Id_{02}) \\
\{\{m,n,r\}_1,s,t\}_2 &= \{\{m,n,r\}_2,s,t\}_2 \\
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Finally, from the identities $(TJM2a), \cdots, (TJM2d)$ we obtain the identities:

$$\begin{aligned}
\{m,n,\{r,s,t\}_1\}_1 =& \{\{m,n,r\}_1,s,t\}_1 - \{r,\{n,m,s\}_2,t\}_2 \\
&+ \{\{m,n,t\}_1,s,r\}_1, && (TJL2a)\\
\{n,m,\{r,s,t\}_1\}_2 =& \{\{n,m,r\}_2,s,t\}_1 - \{r,\{m,n,s\}_1,t)_2 \\
&+ \{\{n,m,t\}_2,s,r\}_1, && (TJL2b)\\
\{\{m,n,r\}_1,s,t\}_1 =& \{\{m,s,t\}_1,n,r\}_1 - \{m,\{n,t,s\}_1,r\}_1 \\
&+ \{m,n,\{r,s,t\}_1\}_1, && (TJM2c)\\
\{\{n,m,r\}_2,s,t\}_1 =& \{\{n,s,t\}_1,m,r\}_2 - \{n,\{m,t,s\}_1,r\}_2 \\
&+ \{n,m,\{r,s,t\}_1\}_2, && (TJM2d)
\end{aligned}$$

Hence, $(M;\{\cdot,\cdot,\cdot\}_1,\{\cdot,\cdot,\cdot\}_2) \in \mathcal{V}(\hat{I})$, with $\hat{I} := \{(Id_{01}),(Id_{02}),(Id_{03})\} \cup \{(TJL1a),(TJL2a),\cdots,(TJL2d)\}$.

Finally, from the identities $(TJM2a), \cdots, (TJM2d)$ we obtain the identities:

$$
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& + \{\{m,n,t\}_1,s,r\}_1, && (TJL2a)\\
\{n,m,\{r,s,t\}_1\}_2 =& \{\{n,m,r\}_2,s,t\}_1 - \{r,\{m,n,s\}_1,t)_2 \\
& + \{\{n,m,t\}_2,s,r\}_1, && (TJL2b)\\
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Hence, $(M; \{\cdot,\cdot,\cdot\}_1, \{\cdot,\cdot,\cdot\}_2) \in \mathcal{V}(\hat{I})$, with
$\hat{I} := \{(Id_{01}), (Id_{02}), (Id_{03})\} \cup \{(TJL1a), (TJL2a), \cdots, (TJL2d)\}$.

Example

The n–ary Loday algebra of a generalized Leibniz n–algebra
let (LD) be the equation

$$[x_1, \cdots x_{n-1}, [y_1, \cdots, y_n]]$$
$$= [[x_1, \cdots x_{n-1}, y_1], y_1, \cdots, y_n] + [y_1, [x_1, \cdots x_{n-1}, y_2], y_3, \cdots, y_n]$$
$$+ \cdots + [y_1, \cdots y_{n-1}, [x_1, \cdots, x_{n-1}, y_n]]. \quad (LD)$$

Sea $I := \{(LD)\}$, and $L \in \mathcal{V}(I)$. This is the **variety of generalized n–ary Leibniz algebras**. M is an n–ary I–module over L if there are multilinear compositions for all $a_1, \cdots, a_n \in A$ and $m \in M$:

$$\begin{array}{ll} M \times L^{n-1} \to M & (m, a_2, \cdots, a_n) \mapsto (m, a_2, \cdots, a_n)_1 \\ L \times M \times L^{n-2} \to M & (a_1, m, a_2, \cdots, a_n) \mapsto (a_1, m, a_2, \cdots, a_n)_2, \cdots, \\ L^{n-1} \times M \to M & (a_1, \cdots, a_{n-1}, m) \mapsto (a_1, \cdots, a_{n-1}, m)_n, \end{array}$$

such that $A \oplus M \in \mathcal{V}(I)$.

So, the bilinear compositions satisfy $2n-1$ independent identities:

$$(m, a_2, \cdots a_{n-1}, [b_1, \cdots, b_n])_1$$
$$= ((m, a_2, \cdots a_{n-1}, b_1)_1, b_2, \cdots, b_n)_1$$
$$+(b_1, (m, a_2, \cdots a_{n-1}, b_1)_1, b_3, \cdots, b_n)_2$$
$$+ \cdots + (b_1, \cdots b_{n-1}, (m, a_2, \cdots a_{n-1}, b_n)_1)_n, \quad (1.1)$$
$$(a_1, m, a_3, \cdots a_{n-1}, [b_1, \cdots, b_n])_2$$
$$= ((a_1, m, a_3, \cdots a_{n-1}, b_1)_2, b_2, \cdots, b_n)_1$$
$$+(b_1, (a_1, m, a_3, \cdots a_{n-1}, b_2)_2, b_3, \cdots, b_n)_2$$
$$+ \cdots + (b_1, \cdots, b_{n-1}, (a_1, m, a_3, \cdots a_{n-1}, b_n)_2)_n, \quad (1.2)$$
$$\vdots$$
$$(a_1, \cdots, a_{n-2}, m, [b_1, \cdots, b_n])_{n-1}$$
$$= ((a_1, \cdots, a_{n-2}, m, b_1)_{n-1}, b_2, \cdots, b_n)_1$$
$$+(b_1, (a_1, \cdots, a_{n-2}, m, b_2)_{n-1}, b_3, \cdots, b_n)_2$$
$$+ \cdots + (b_1, \cdots b_{n-1}, (a_1, \cdots, a_{n-2}, m, b_n)_{n-1})_n, \quad (1.n)$$

Observe that each identity $(1, j)$ has a term with a unique parenthesis choice (no other identity has it) with outer bilinear composition $(\centerdot, \cdots, \centerdot)_j$ with a bilinear composition $(\centerdot, \cdots, \centerdot)_j$ in its j–th entry.

and the identities:

$$(a_1, a_2, \cdots a_{n-1}, (m, b_2, \cdots, b_n)_1)_n$$
$$= ((a_1, \cdots a_{n-1}, m)_n, b_2, \cdots, b_n)_1 + (m, [a_1, \cdots a_{n-1}, b_1], b_3, \cdots, b_n)_1$$
$$+ \cdots + (m, b_1, \cdots b_{n-1}, [a_1, a_2, \cdots a_{n-1}, b_n])_1, \quad (2.1)$$
$$(a_1, a_2, \cdots a_{n-1}, (b_1, m, b_2, \cdots, b_n)_2)_n$$
$$= ([a_1, \cdots a_{n-1}, b_1], m, b_3, \cdots, b_n)_2 + (b_1, (a_1, \cdots a_{n-1}, m)_n, b_2, \cdots, b_n)_2$$
$$+ (b_1, m, [a_1, \cdots a_{n-1}, b_3], b_4, \cdots, b_n)_2 + \cdots +$$
$$(b_1, m, b_3, \cdots b_{n-1}, [a_1, a_2, \cdots a_{n-1}, b_n])_2, \quad (2.2)$$
$$\vdots$$
$$(a_1, a_2, \cdots a_{n-1}, (b_1, b_2, \cdots, b_{n-1}, m)_n)_n$$
$$= ([a_1, \cdots a_{n-1}, b_1], b_2, \cdots, b_{n-1}, m)_n) +$$
$$(b_1, [a_1, \cdots a_{n-1}, b_2], b_3, \cdots, b_{n-1}, m)_n$$
$$+ \cdots + (b_1, b_2, \cdots, b_{n-2}, [a_1, \cdots a_{n-1}, b_{n-1}], m)_n +$$
$$(b_1, b_2, \cdots, b_{n-1}, (a_1, \cdots a_{n-1}, m)_n)_n. \quad (2.n)$$

Observe that each identity $(2, j)$ has a term with a unique parenthesis choice (no other identity has it) with outer bilinear composition $(\centerdot, \cdots, \centerdot)_j$ with a bilinear composition $(\centerdot, \cdots, \centerdot)_n$ in its j–th entry.

Hence, all the identities are independent.

Consider now a surjective equivariant function $\xi : M \to L$ such that for all $a_1, \cdots, a_n \in L$, and $m \in M$:
$\xi((m, a_2, \cdots, a_n)_1) = [\xi(m), a_2, \cdots, a_n]$,
$\xi((a_1, m, a_3, \cdots, a_n)_2) = [a_1, \xi(m), a_3, \cdots, a_n], \cdots,$
$\xi((a_1, a_2, \cdots, a_{n-1}, m)_1) = [a_1, a_2, \cdots, a_{n-1}, \xi(m)]$.
We define the n-ary products in M:
$[m_1, m_2, \cdots, m_n]_1 = (m_1, \xi(m_2), \cdots, \xi(m_n))_1,$
$[m_1, m_2, \cdots, m_n]_2 = (\xi(m_1), m_2, \xi(m_3), \cdots, \xi(m_n))_2, \cdots,$
$[m_1, m_2, \cdots, m_{n-1}, m_n]_n = (\xi(m_1), \xi(m_2), \cdots, \xi(m_{n-1}), m_n)_n.$
From the equivariance of ξ we obtain the 0–th identities:
$[m_1, \cdots, m_{t-1}, [\eta_1, \cdots, \eta_n]_r, m_{t+1}, \cdots, m_n]_i =$
$[m_1, \cdots, m_{t-1}, [\eta_1, \cdots, \eta_n]_s, m_{t+1}, \cdots, m_n]_n,$
$\forall t \neq i; \quad t, i, r, s \in \{1, \cdots, n\}.$

From the identities $(1,j); j=1,\cdots,n-1$, we obtain the identities:

$$[m_1,m_2,\cdots m_{n-1},[\eta_1,\cdots,\eta_n]_1]_1$$
$$=[[m_1,m_2,\cdots m_{n-1},\eta_1]_1,\eta_2,\cdots,\eta_n]_1+$$
$$[\eta_1,[m_1,m_2,\cdots m_{n-1},\eta_2]_1,\eta_3,\cdots,\eta_n]_2$$
$$+\cdots+[\eta_1,\cdots\eta_{n-1},[m_1,m_2,\cdots m_{n-1},\eta_n]_1]_n, \quad (1a.1)$$
$$[m_1,m_2,\cdots m_{n-1},[\eta_1,\cdots,\eta_n]_1]_2$$
$$=[[m_1,m_2,\cdots m_{n-1},\eta_1]_2,\eta_2,\cdots,\eta_n]_1+$$
$$[\eta_1,[m_1,m_2,\cdots m_{n-1},\eta_2]_2,\eta_3,\cdots,\eta_n]_2$$
$$+\cdots+[\eta_1,\cdots\eta_{n-1},[m_1,m_2,\cdots m_{n-1},\eta_n]_2]_n, \quad (1a.2)$$
$$\vdots$$
$$[[m_1,\cdots,m_{n-1},[\eta_1,\cdots,\eta_n]_1]_{n-1}$$
$$=[[m_1,\cdots,m_{n-1},\eta_1]_{n-1},\eta_2,\cdots,\eta_n]_1+$$
$$[\eta_1,[m_1,\cdots,m_{n-1},\eta_2]_{n-1},\eta_3,\cdots,\eta_n]_2$$
$$+\cdots+[\eta_1,\cdots\eta_{n-1},[m_1,\cdots,m_{n-1},\eta_n]_{n-1}]_n. \quad (1a.n-1)$$

Observe that each identity $(1a,j)$ has a term with a unique parenthesis choice (no other identity has it) with outer n–ary product $[\centerdot,\cdots,\centerdot]_j$ with a n–ary product $[\centerdot,\cdots,\centerdot]_j$ in its j–th entry.

From the identities $(2,j); j = 1,\cdots,n$, we obtain the identities:

$$[m_1, m_2, \cdots m_{n-1}, [\eta_1, \cdots, \eta_n]_1]_n$$
$$= [[m_1, m_2, \cdots m_{n-1}, \eta_1]_n, \eta_2, \cdots, \eta_n]_1 +$$
$$[\eta_1, [m_1, m_2, \cdots m_{n-1}, \eta_2]_1, \eta_3, \cdots, \eta_n]_1$$
$$+ \cdots + [\eta_1, \cdots \eta_{n-1}, [m_1, m_2, \cdots m_{n-1}, \eta_n]_1]_1, \quad (2a.1)$$
$$[m_1, m_2, \cdots m_{n-1}, [\eta_1, \cdots, \eta_n]_2]_n$$
$$= [[m_1, m_2, \cdots m_{n-1}, \eta_1]_1, \eta_2, \cdots, \eta_n]_2 +$$
$$[\eta_1, [m_1, m_2, \cdots m_{n-1}, \eta_2]_n, \eta_3, \cdots, \eta_n]_2$$
$$+[\eta_1, \eta_2, [m_1, m_2, \cdots m_{n-1}, \eta_3]_1, \eta_4, \cdots, \eta_n]_2 + \cdots +$$
$$[\eta_1, \cdots \eta_{n-1}, [m_1, m_2, \cdots m_{n-1}, \eta_n]_1]_2, \quad (2a.2)$$
$$\vdots$$
$$[m_1, \cdots, m_{n-1}, [\eta_1, \cdots, \eta_n]_n]_n$$
$$= [[m_1, \cdots, m_{n-1}, \eta_1]_1, \eta_2, \cdots, \eta_n]_n + [\eta_1, [m_1, \cdots, m_{n-1}, \eta_2]_1, \eta_3, \cdots, \eta_n]_n$$
$$+ \cdots + [\eta_1, \cdots, \eta_{n-2}, [m_1, \cdots, m_{n-1}, \eta_{n-1}]_1, \eta_n]_n +$$
$$[\eta_1, \cdots \eta_{n-1}, [m_1, \cdots, m_{n-1}, \eta_n]_n]_n, \quad (2a.n)$$

Observe that each identity $(2a,j)$ has a term with a unique parenthesis choice (no other identity has it) with outer n–ary product $[\centerdot, \cdots, \centerdot]_j$ with a n–ary product $[\centerdot, \cdots, \centerdot]_n$ in its j–th entry.

Hence, the $2n-1$ identities ($(1a,j)$ for $j=1,\cdots,n-1$ and $(2a,j)$ for $j=1,\cdots,n$) are independent, and
$\hat{I}$:= set of 0-th identities union $\{(1a,j)|j=1,\cdots,n-1\}$ union $\{(2a,j)|j=1,\cdots,n\}$ is set of identities fulfilled by $(M;[\centerdot,\cdots,\centerdot]_1,\cdots,[\centerdot,\cdots,\centerdot]_n)$.