Algebras with operators, Koszul duality, and conformal algebras

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In this talk we will observe and discuss relations between several classes of algebraic systems that appeared in areas including algebraic topology, homological algebra, K-theory, combinatorics, and mathematical physics. A brief sketch of the picture is shown in the table below.

Conformal algebras				(?)		
$\vert$				$\vert$		
Algebras & averaging operators				Algebras & Rota–Baxter operators	—	Classical Yang–Baxter equations
$\vert$				$\vert$		
$\mathcal{O} \circ$ Perm	—	Operad $\mathcal{O}$	—	$\mathcal{O} \bullet$ pre-Lie		
$\vert$				$\vert$		
Loday algebras (dialgebras)	—	**Algebras**	—	**Dendriform algebras (pre-algebras)**		
$\vert$				$\vert$		
Perm (left-comm.)	—	Com	—	Zinbiel (dual to Leibniz)		
Leibniz	—	Lie	—	pre-Lie (left-symmetric)		

There are three main columns that can be informally labeled as related to the classes of “ordinary” algebras (central column), Loday algebras also called dialgebras (left column), and dendriform algebras (right column). The left and the right columns present two natural ways of “replicating” the operads governing the classes of ordinary algebras, let us call them both as replicated classes.

As a model example one may consider the class Lie of all Lie algebras: The corresponding replicated classes of algebras are known as Leibniz and pre-Lie (also called left-symmetric) algebras.

In the binary quadratic case, one may observe Koszul duality between the left and right columns, which is very natural since the corresponding replicated operads may be obtained as white and black Manin products with Koszul-dual operads Perm (governing the class of left-commutative algebras) and pre-Lie, respectively. We observe a general method how to deduce the identities defining the replicated classes for an arbitrary operad, not necessarily binary or quadratic.

Further, we consider a relation between algebras in these replicated classes and ordinary algebras with additional structures. The idea of Koszul duality works here and provides a way to embed a Loday algebra into an ordinary algebra with an averaging operator, i.e., a linear map T such that

$$T(x)T(y) = T(xT(y)) = T(T(x)y).$$

Similarly, a dendriform algebra may be embedded into an ordinary algebra with a Rota–Baxter operator, i.e., a linear map R such that

$$R(x)R(y) = R(xR(y)) + R(R(x)y).$$

The right column of the table seems to be more interesting from the algebraic point of view. For example, pre-Lie algebras (dendriform Lie algebras) have an extremely wide area of applications, Rota–Baxter operators appeared initially in fluctuation theory (by G. Baxter) and later in probability theory (G.-C. Rota), they turn to be closely related with the classical Yang–Baxter equation (after C. Yang and R. Baxter). This is why the algebraic study of dendriform algebras and related structures is an actual and even urgent topic of research.

In contrast, the left column (Leibniz algebras and around) have less applications. Notably, there is a general way how to solve algebraic problems for these structures by means of conformal algebras. The latter were introduced by V. Kac as a tool in the study of vertex operator algebras: The singular part of an operator product expansion describes the commutator of two chiral fields in 2-dimensional conformal field theory. Every Loday algebra may be naturally embedded into an appropriate conformal algebra. This observation allows to reduce many problems concerning this replication to the case of ordinary algebras. Unfortunately, a "dual analogue" of conformal algebras, which may work in the same way for dendriform algebras, is unknown.

As an example, we consider analogues of the Poincare–Birkhoff–Witt and Ado theorems for Lie algebras. These statements are known to be true for Leibniz algebras but remain unclear in the dendriform settings (for pre-Lie algebras).