A Magic Pyramid of Supergravities

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based on

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Aims

- MATHEMATICS:

  $$(DIVISION\ ALGEBRAS)^2 = MAGIC\ SQUARE\ OF\ LIE\ ALGEBRAS$$

- PHYSICS (in D=3):

  $$(YANG - MILLS)^2 = MAGIC\ SQUARE\ OF\ SUPERGRAVITIES$$

- RESULT :

  $$MATHEMATICS\ MAGIC\ SQUARE = PHYSICS\ MAGIC\ SQUARE$$

- PHYSICS (in D=3,4,6,10):

  $$(YANG - MILLS)^2 = MAGIC\ PYRAMID\ OF\ SUPERGRAVITIES$$

Outline

- 1) Physics background
- 2) Division Algebras
- 3) Super-Yang-Mills
- 4) Squaring Yang-Mills
- 5) Magic square of supergravities in D=3
- 6) Magic pyramid of supergravities in D=3, 4, 6, 10

1.1 Physics background

- Strong nuclear, Weak nuclear and Electromagnetic forces described by Yang-Mills gauge theory, a non-abelian generalisation of Maxwell theory with vector field A_μ. Gluons, W, Z and photons have spin 1.

- Gravitational force described by Einstein's general relativity with metric tensor $g_{\mu\nu}$. Gravitons have spin 2.

- But recent work suggests maybe $(Gravity) = (Yang - Mills)^2$

- Today focus on deriving global symmetries G of supergravity from product of two super-Yang-Mills theories eg in D=4

  $(N = 4 \text{ YM}) \times (N = 4 \text{ YM}) \rightarrow (N = 8 \text{ supergravity with } G = E_7)$

  N=number of supersymmetries

1.2 Supersymmetry

- Special relativity requires that laws of physics be invariant under the translations P_μ and Lorentz rotations $M_{\mu\nu}$ ($\mu = 0, 1, 2, 3$) of the Poincare algebra

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}$$

  Representations are either bosons (integer spin) or fermions (odd half-integer spin).

- The Super-Poincare algebra incorporates anticommuting spin 1/2 changing transformations Q_α

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma_\mu)_{\alpha\dot{\beta}} P_\mu$$

$$[Q_\alpha, P_\mu] = 0 \qquad [M_{\mu\nu}, Q_\alpha] = (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta$$

  Representations mix bosons and fermions (equal number).

1.3. Generalizations to $D \neq 4$ and $N > 1$

- Global: eg super-Yang-Mills. Note D=3,4,6,10 are special

D	N							
10	1							
9	1							
8	1							
7	1							
6	1	2						
5	1	2						
4	1	2	×	4				
3	1	2	3	4	5	6	×	8

- Local: supergravity

D	N															
11	1															
10	1	2														
9	1	2														
8	1	2														
7	1	2														
6	1	2	3	4												
5	1	2	3	4												
4	1	2	3	4	5	6	×	8								
3	1	2	3	4	5	6	7	8	9	10	11	12	13	14	×	16

2.2 Division algebras

- Normed Divison Algebras: four kinds of numbers for which the Octonions occupy a privileged position:

Name	Symbol	Imaginary parts
Reals	$\mathbb{R}$	0
Complexes	$\mathbb{C}$	1
Quaternions	$\mathbb{H}$	3
Octonions	$\mathbb{O}$	7

Table: Division Algebras

2.3 Division algebras

- Division: $ax + b = 0$ has a unique solution
- Associative: $a(bc) = (ab)c$
- Commutative: $ab = ba$

A	construction	division?	associative?	commutative?	ordered?
$\mathbb{R}$	R	yes	yes	yes	yes
$\mathbb{C}$	$R + e_1 R$	yes	yes	yes	no
$\mathbb{H}$	$C + e_2 C$	yes	yes	no	no
$\mathbb{O}$	$H + e_3 H$	yes	no	no	no
S	$O + e_4 O$	no	no	no	no

(Note that Math cutoff at O resembles Physics cutoff at N=8)

- They provide an intuitive basis for the exceptional Lie algebras:

Classical algebras		Rank	Dimension
A_n	$SU(n+1)$	n	$(n+1)^2-1$
B_n	$SO(2n+1)$	n	$n(2n+1)$
C_n	$Sp(2n)$	n	$n(2n+1)$
D_n	$SO(2n)$	n	$n(2n-1)$
Exceptional algebras			
E_6		6	78
E_7		7	133
E_8		8	248
F_4		4	52
G_2		2	14

Table: Classical and exceptional Lie alebras

2.5 Magic square

- Freudenthal-Rozenfeld-Tits magic square

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 + A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Table: Magic square

2.6 Octonions

For a review see [Baez:2001].

- An element $x \in \mathbb{O}$ may be written $x = x^a e_a$, where $a = 0, \ldots, 7$, $x^a \in \mathbb{R}$ and $\{e_a\}$ is a basis with one real $e_0 = 1$ and seven $e_i, i = 1, \ldots, 7$ imaginary elements. The octonionic conjugation is denoted by e_a^*, where $e_0^* = e_0$ and $e_i^* = -e_i$.
- The octonionic multiplication rules are

$$e_0^2 = 1 \qquad e_0 e_i = e_i \qquad e_i e_j = Cijk e_k,$$

where the totally antisymmetric C_{ijk} are the octonionic structure constants:

$$C_{ijk} = \varepsilon_{ijk} \text{ if } ijk \in \mathbf{L}$$

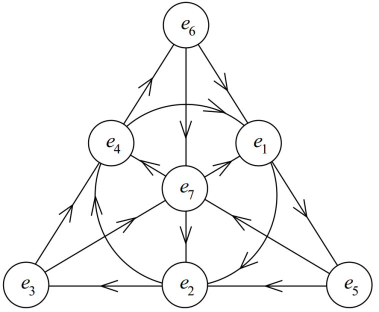
with $\mathbf{L}$ the set of oriented lines of the Fano plane.

$$\mathbf{L} = \{124, 235, 346, 457, 561, 672, 713\}.$$

2.7 Fano plane

The Fano plane has seven points and seven lines (the circle counts as a line) with three points on every line and three lines through every point.

Fano plane



2.8 Gino Fano

Gino Fano (5 January 1871 to 8 November 1952) was an Italian mathematician. He was born in Mantua and died in Verona.
Fano worked on projective and algebraic geometry; the Fano plane, Fano fibration, Fano surface, and Fano varieties are named for him.
Ugo Fano and Robert Fano were his sons.

2.9 Fano quadrangles

- It will also be useful to define Q_{ijkl}, which is equal to 1 (-1) when $ijkl$ is an even (odd) permutation of an element of $\mathbf{Q}$, the set of oriented quadrangles in the Fano plane:

$$\mathbf{Q} = \{3567, 4671, 5712, 6123, 7234, 1345, 2456\},$$

and equal to zero otherwise. Equivalently, we can define Q_{ijkl} by

$$Q_{ijkl} = -\frac{1}{3!} C_{mnp} \varepsilon_{mnpijkl}.$$

2.10 The Associator

- The octonions have a trilinear map called the associator given by :

$$[x, y, z] = (xy)z - x(yz)$$

which measures the failure of associativity.

- In the same way that the multiplication of the octonionic bases was realised using the lines of the Fano plane, the associator of three octonionic bases can be realised using its quadrangles **Q** :

$$[e_a, e_b, e_c] = 2Q_{abcd}e_d$$

where $e_a = (e_0, e_i)$. The object Q_{abcd} is totally antisymmetric with $Q_{0ijk} = C_{ijk}$.

$$Q_{abcd} = \frac{1}{4!}\epsilon_{abcdefgh}Q^{efgh}$$

2.11 Norm-preserving algebras

- To understand the symmetries of the magic square and its relation to YM we shall need in particular two algebras defined on $\mathbb{A}$.
- First, the algebra $\mathfrak{norm}(\mathbb{A})$ that preserves the norm

$$\langle x|y\rangle := \frac{1}{2}(x\overline{y} + y\overline{x}) = x^a y^b \delta_{ab}$$

$$\mathfrak{norm}(\mathbb{R}) = 0$$
$$\mathfrak{norm}(\mathbb{C}) = \mathfrak{so}(2)$$
$$\mathfrak{norm}(\mathbb{H}) = \mathfrak{so}(4)$$
$$\mathfrak{norm}(\mathbb{O}) = \mathfrak{so}(8)$$

2.12 Triality Algebra

- Second, the triality algebra $\mathfrak{tri}(\mathbb{A})$

$$\mathfrak{tri}(\mathbb{A}) := \{(A, B, C)|A(xy) = B(x)y + xC(y)\}, \quad A, B, C \in \mathfrak{so}(n), \quad x, y \in \mathbb{A}\}$$

$$\begin{aligned}
\mathfrak{tri}(\mathbb{R}) &= 0 \\
\mathfrak{tri}(\mathbb{C}) &= \mathfrak{so}(2) + \mathfrak{so}(2) \\
\mathfrak{tri}(\mathbb{H}) &= \mathfrak{so}(3) + \mathfrak{so}(3) + \mathfrak{so}(3) \\
\mathfrak{tri}(\mathbb{O}) &= \mathfrak{so}(8)
\end{aligned}$$

[Barton and Sudbery:2003]:

2.13 Magic square of (non-compact) algebras $\mathfrak{g}$

- For the purposes of squaring YM a manifestly $\mathbb{A}_L \leftrightarrow \mathbb{A}_R$ symmetric formulation of the square is required.
- This is achieved by adapting the *triality algebra* construction introduced by Barton and Sudbery.
[Barton and Sudbery:2003, MJD et al:2013]
- Our definition is given by (see [MJD et al:2013] for commutators):

$$\mathfrak{g}_3(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) + \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \times \mathbb{A}_R).$$

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SL(2,\mathbb{R})$	$SU(2,1)$	$USp(4,2)$	$F_{4(-20)}$
$\mathbb{C}$	$SU(2,1)$	$SU(2,1)\times SU(2,1)$	$SU(4,2)$	$E_{6(-14)}$
$\mathbb{H}$	$USp(4,2)$	$SU(4,2)$	$SO(8,4)$	$E_{7(-5)}$
$\mathbb{O}$	$F_{4(-20)}$	$E_{6(-14)}$	$E_{7(-5)}$	$E_{8(8)}$

$$\mathfrak{e}_{8(8)} = \mathfrak{so}(\mathbb{O}) + \mathfrak{so}(\mathbb{O}) + 3(\mathbb{O}\times\mathbb{O})$$

$$248 = (28,1) + (1,28) + (8_v,8_v) + (8_s,8_s) + (8_c,8_c)$$

2.14 Comment

- There are different magic squares depending on the choice of real forms.

- Ours should not to be confused with other versions that have appeared in the so-called "magic" supergravities in $D = 4, 5, 6$ [Gunaydin, Sierra and Townsend 1983]

- The specific square of real forms we derive from the triality construction was first obtained in [Cacciatori-Cerchiai-Marrani:2012] using a "Lorentizian Jordan algebra" adaptation of the Tits formula [Tits:1962]

2.15 Magic square of maximal compact subalgebras H

- We shall also need a magic square of the maximal compact subalgebras. This is given by the *reduced* triality construction,

$$\mathfrak{g}_1(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) + \mathfrak{tri}(\mathbb{A}_R) + (\mathbb{A}_L \times \mathbb{A}_R),$$

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	SO(2)	SO(3) $\times$ SO(2)	SO(5) $\times$ SO(3)	SO(9)
$\mathbb{C}$	SO(3) $\times$ SO(2)	[SO(3) $\times$ SO(2)]^2	SO(6) $\times$ SO(3)	SO(10) $\times$ SO(2)
$\mathbb{H}$	SO(5) $\times$ SO(3)	SO(6) $\times$ SO(3)	SO(8) $\times$ SO(4)	SO(12) $\times$ SO(3)
$\mathbb{O}$	SO(9)	SO(10) $\times$ SO(2)	SO(12) $\times$ SO(3)	SO(16)

Table: Magic square of maximal compact subalgebras.

3.0 Yang-Mills

- Lie-algebra valued 1-form A

  Covariant derivative $DX = dX + [A, X]$

  Commutator $DDX = [F, X]$

  2-form field strength $F = dA + AA$

  Bianchi identity $DF = 0$

  Field equation $D * F = 0$

  Action principle $S = \frac{1}{2} \int Tr(F * F)$

3.1 Supersymmetry

- We give a unified description of
  $D = 3$ Yang-Mills with $\mathcal{N} = 1, 2, 4, 8$
  $D = 4$ Yang-Mills with $\mathcal{N} = 1, 2, 4$
  $D = 6$ Yang-Mills with $\mathcal{N} = 1, 2$
  $D = 10$ Yang-Mills with $\mathcal{N} = 1$
  in terms of a pair of division algebras $(\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}})$, $n = D - 2$
- We present a master Lagrangian, defined over $\mathbb{A}_{n\mathcal{N}}$-valued fields, which encapsulates all cases.
- The overall (spacetime plus internal) on-shell symmetries are given by the corresponding *triality* algebras.
- We use imaginary $\mathbb{A}_{n\mathcal{N}}$-valued auxiliary fields to close the non-maximal supersymmetry algebra off-shell. The failure to close off-shell for maximally supersymmetric theories is attributed directly to the non-associativity of the octonions.
  NB This is not applicable to $D = 10$, so $D < 10$ Yang-Mills not just "trivial" dimensional reduction of $D = 10$.

3.2 Earlier work

- For earlier work on $\mathbb{A} = (\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$ and $D = (3, 4, 6, 10)$
  [Kugo and Townsend: 1983
  Sudbery: 1984
  Evans: 1987, 1994
  MJD: 1987
  Fairlie and Manogue: 1987
  Manogue and Schray: 1993
  Baez: 2001, 2009
  Baez-Huerta 2009-2017: "Division algebras and supersymmetry I-IV"
  Anastasiou, Borsten, Duff, Hughes, Marrani, Nagy 2014-2017]
- For other work on octonions in high energy physics:
  [Gunaydin and Gursey: 1973, 1974
  Gunaydin, Sierra and Townsend: 1983
  Fubini and Nicolai: 1985
  Berkovits: 1993
  Gunaydin and Nicolai:1995
  Dray and Manogue: 2004
  Feingold, Kleinschmidt and Nicolai: 2008]

3.3 $D = 3, \mathcal{N} = 8$ Yang-Mills

- The $D = 3$, $\mathcal{N} = 8$ super YM action is given by

$$S = \int d^3x \left(-\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} - \frac{1}{2} D_\mu \phi^A_i D^\mu \phi^A_i + i\bar{\lambda}^A_a \gamma^\mu D_\mu \lambda^A_a \right.$$
$$\left. -\frac{1}{4} g^2 f_{BC}{}^A f_{DE}{}^A \phi^B_i \phi^D_i \phi^C_j \phi^E_j - g f_{BC}{}^A \phi^B_i \bar{\lambda}^{Aa} \Gamma^i_{ab} \lambda^{Cb} \right),$$

  where the Dirac matrices Γ^i_{ab}, $i = 1, \ldots, 7$, $a, b = 0, \ldots, 7$, belong to the SO(7) Clifford algebra.
- The key observation is that Γ^i_{ab} can be represented by the octonionic structure constants,

$$\Gamma^i_{ab} = i(\delta_{bi}\delta_{a0} - \delta_{b0}\delta_{ai} + C_{iab}),$$

  which allows us to rewrite the action over octonionic fields

3.4 $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \to \mathcal{N} = 1, 2, 4, 8$ supersymmetries

- If we replace $\mathbb{O}$ with a general division algebra $\mathbb{A}$, the result is $\mathcal{N} = 1, 2, 4, 8$ over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$:

$$\begin{aligned} S = \int d^3x \Big(& -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} - \frac{1}{2} D_\mu \phi^{*A} D^\mu \phi^A + i\bar{\lambda}^A \gamma^\mu D_\mu \lambda^A \\ & -\frac{1}{4} g^2 f_{BC}{}^A f_{DE}{}^A \langle \phi^B | \phi^D \rangle \langle \phi^C | \phi^E \rangle \\ & +\frac{i}{2} g f_{BC}{}^A \left((\bar{\lambda}^A \phi^B) \lambda^C - \bar{\lambda}^A (\phi^{*B} \lambda^C) \right) \Big), \end{aligned}$$

- $\phi = \phi^i e_i$ is an Im$\mathbb{A}$-valued scalar field.
- $\lambda = \lambda^a e_a$ is an $\mathbb{A}$-valued two-component spinor and $\bar{\lambda} = \bar{\lambda}^a e_a^*$.
- Note, since λ^a is anti-commuting we are dealing with the *algebra of octonions defined over the Grassmanns*.

3.5 Transformation rules

- The supersymmetry transformations in this language are given by

$$\begin{aligned}
\delta\lambda^A &= \frac{1}{2}(F^{A\mu\nu} + \varepsilon^{\mu\nu\rho} D_\rho \phi^A)\sigma_{\mu\nu}\epsilon + \frac{1}{2} g f_{BC}{}^A \phi_i^B \phi_j^C \sigma_{ij}\epsilon, \\
\delta A_\mu^A &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu \lambda^A - \bar{\lambda}^A \gamma_\mu \epsilon), \\
\delta\phi^A &= \frac{i}{2} e_i[(\bar{\epsilon} e_i)\lambda^A - \bar{\lambda}^A (e_i \epsilon)],
\end{aligned}$$

  where $\sigma_{\mu\nu}$ are the generators of $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SO}(1,2)$.

- For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we can add an auxiliary $\mathbb{A}$-valued scalar for an off-shell formulation of the supersymmetry algebra.

- For $\mathbb{A} = \mathbb{O}$ the algebra fails to close because of non-associativity. [MJD et al: 2013]

4.3 Squaring $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ Yang-Mills

We have:

- Cast the magic square in terms of a manifestly $\mathbb{A}_L \leftrightarrow \mathbb{A}_R$ symmetric triality algebra construction,
- Written $\mathcal{N} = 1, 2, 4, 8$ YM in terms of fields valued in $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

We can now:

- Obtain the magic square of supergravities by "squaring" $\mathcal{N} = 1, 2, 4, 8$ YM.
- In the supersymmetric context it is not difficult to see that the amount of supersymmetry is given by

$$[\mathcal{N}_L \text{ SYM}] \times [\mathcal{N}_R \text{ SYM}] \rightarrow [\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R \text{ SG}],$$

4.4 U-dualities of supergravity

- It is harder to see how the other gravitational symmetries arise from those of Yang-Mills.
- For example, $D = 4, N = 8$ supergravity has a global non-compact symmetry (U-duality) $E_{7(7)}$ and a local compact symmetry $SU(8)$, but $D = 4, N = 4$ super Yang-Mills has global $SU(4)$ R-symmetry.

$\mathcal{N}$	D	scalars	vectors	G	H
2	10A	1	1	$SO(1,1,\mathbb{R})$	$-$
4	6	25	16	$SO(5,5,\mathbb{R})$	$SO(5,\mathbb{R}) \times SO(5,\mathbb{R})$
8	4	70	28	$E_{7(7)}(\mathbb{R})$	$SU(8)$
16	3	128	-	$E_{8(8)}(\mathbb{R})$	$SO(16,\mathbb{R})$

Table: Symmetry groups (G) of maximal supergravity in D dimensions and their maximal compact subgroups (H). One may truncate to lower $\mathcal{N}$ to get smaller G and H. The scalars belong to the space G/H.

4.5. Squaring $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ Yang-Mills in $D = 3$

Taking a left SYM multiplet

$$\{A_\mu(L) \in \mathrm{Re}\mathbb{A}_L, \quad \phi(L) \in \mathrm{Im}\mathbb{A}_L, \quad \lambda(L) \in \mathbb{A}_L\}$$

and tensoring with a right multiplet

$$\{A_\mu(R) \in \mathrm{Re}\mathbb{A}_R, \quad \phi(R) \in \mathrm{Im}\mathbb{A}_R, \quad \lambda(R) \in \mathbb{A}_R\}$$

we obtain the field content of a supergravity theory valued in both $\mathbb{A}_L$ and $\mathbb{A}_R$:

$\mathbb{A}_L/\mathbb{A}_R$	$A_\mu(R) \in \mathrm{Re}\mathbb{A}_R$	$\phi(R) \in \mathrm{Im}\mathbb{A}_R$	$\lambda(R) \in \mathbb{A}_R$
$A_\mu(L) \in \mathrm{Re}\mathbb{A}_L$	$g_{\mu\nu} + \varphi \in \mathrm{Re}\mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\varphi \in \mathrm{Re}\mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\Psi_\mu + \chi \in \mathrm{Re}\mathbb{A}_L \otimes \mathbb{A}_R$
$\phi(L) \in \mathrm{Im}\mathbb{A}_L$	$\varphi \in \mathrm{Im}\mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\varphi \in \mathrm{Im}\mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\chi \in \mathrm{Im}\mathbb{A}_L \otimes \mathbb{A}_R$
$\lambda(L) \in \mathbb{A}_L$	$\Psi_\mu + \chi \in \mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\chi \in \mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\varphi \in \mathbb{A}_L \otimes \mathbb{A}_R$

Grouping spacetime fields of the same type we find,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}, \quad \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}$$

4.6 Grouping together

- Grouping spacetime fields of the same type we find,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi, \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}. \tag{1}$$

- Note we have dualised all resulting p-forms, in particular vectors to scalars.
- The $\mathbb{R}$-valued graviton and $\mathbb{A}_L \oplus \mathbb{A}_R$-valued gravitino carry no degrees of freedom.
- The $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$-valued scalar and Majorana spinor each have $2(\dim \mathbb{A}_L \times \dim \mathbb{A}_R)$ degrees of freedom.
- Fortunately, $\mathbb{A}_L \oplus \mathbb{A}_R$ and $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ are precisely the representation spaces of the vector and (conjugate) spinor.
- E.g. in the maximal case of $\mathbb{A}_L, \mathbb{A}_R = \mathbb{O}$, we have the $\mathbf{16}, \mathbf{128}$ and $\mathbf{128}'$ of SO(16).

4.7. Gravity Magic Square in $D = 3$

- In $D = 3$ tensoring $\mathcal{N}_L = 1, 2, 4, 8$ and $\mathcal{N}_R = 1, 2, 4, 8$ Yang-Mills multiplets yields a 4×4 square description of supergravities with $\mathcal{N} = 2, 3, 4, 5, 6, 8, 9, 10, 12, 16$.

$$\begin{array}{cccc} \mathbb{RR} & \mathbb{RC} & \mathbb{RH} & \mathbb{RO} \\ \mathbb{CR} & \mathbb{CC} & \mathbb{CH} & \mathbb{CO} \\ \mathbb{HR} & \mathbb{HC} & \mathbb{HH} & \mathbb{HO} \\ \mathbb{OR} & \mathbb{OC} & \mathbb{OH} & \mathbb{OO} \end{array} \qquad \begin{array}{cccc} 2 & 3 & 5 & 9 \\ 3 & 4 & 6 & 10 \\ 5 & 6 & 8 & 12 \\ 9 & 10 & 12 & 16 \end{array}$$

- The G/H U-dualities are precisely those of the Freudenthal Magic Square!

$$G: \quad \mathfrak{g}_3(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) + \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \times \mathbb{A}_R).$$

$$H: \quad \mathfrak{g}_1(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) + \mathfrak{tri}(\mathbb{A}_R) + (\mathbb{A}_L \times \mathbb{A}_R).$$

4.8 U-dualities from division algebras

- U-dualities G are realised non-linearly on the scalars, which parametrise the symmetric spaces G/H.
- This can be understood using the remarkable identity:

$$(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2 \cong G/H.$$

  For more on these "projective planes" see e.g. [Rosenfeld: 1954, 1995, Freudenthal:1964, Baez: 2001, Landsberg and Manivel: 2001]
- The scalar fields may be regarded as points in $(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2$
- The tangent space at any point of $(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2$ is just $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$, the required representation space of H
- Example: the *Cayley plane* $\mathbb{O}\mathbb{P}^2$, with isometry group $F_{4(-52)}$:

$$F_{4(-52)}/\mathrm{Spin}(9) \cong (\mathbb{R} \otimes \mathbb{O})\mathbb{P}^2 = \mathbb{O}\mathbb{P}^2$$

  The tangent space at any point of $\mathbb{O}\mathbb{P}^2$ is $\mathbb{O}^2$, the spinor of Spin(9)

4.9 Magic square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathcal{N}=2, f=4$ $G=\mathrm{SL}(2,\mathbb{R})$, dim 3 $H=\mathrm{SO}(2)$, dim 1	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36
$\mathbb{C}$	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=4, f=16$ $G=\mathrm{SU}(2,1)^2$, dim 16 $H=\mathrm{SU}(2)^2\times\mathrm{SO}(2)^2$, dim 8	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46
$\mathbb{H}$	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=8, f=64$ $G=\mathrm{SO}(8,4)$, dim 66 $H=\mathrm{SO}(8)\times\mathrm{SO}(4)$, dim 34	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69
$\mathbb{O}$	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69	$\mathcal{N}=16, f=256$ $G=E_{8(8)}$, dim 248 $H=\mathrm{SO}(16)$, dim 120

- The $\mathcal{N}>8$ supergravities in $D=3$ are unique, all fields belonging to the gravity multiplet, while those with $\mathcal{N}\leq 8$ may be coupled to k additional matter multiplets [Marcus and Schwarz:1983; deWit, Tollsten and Nicolai:1992]. The real miracle is that tensoring left and right YM multiplets yields the field content of $\mathcal{N}=2,3,4,5,6,8$ supergravity with $k=1,1,2,1,2,4$: just the right matter content to produce the U-dualities appearing in the magic square.

4.10 Conclusion

- In both cases the field content is such that the U-dualities exactly match the algebras of the magic square:

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SL(2,\mathbb{R})$	$SU(2,1)$	$USp(4,2)$	$F_{4(-20)}$
$\mathbb{C}$	$SU(2,1)$	$SU(2,1)\times SU(2,1)$	$SU(4,2)$	$E_{6(-14)}$
$\mathbb{H}$	$USp(4,2)$	$SU(4,2)$	$SO(8,4)$	$E_{7(-5)}$
$\mathbb{O}$	$F_{4(-20)}$	$E_{6(-14)}$	$E_{7(-5)}$	$E_{8(8)}$

Table: Magic square

- This $D = 3$ square is the base of a Magic Pyramid

5.0 D=3,4,6,10

- BUT there is also a more familiar $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ description of spacetime based on

$$SL(2, A) = SO(n+1, 1)$$

  the Lorentz algebra in $D = n + 2$ dimensions (not strictly true when A=O but Sudbery has a way of dealing with it.)

- Baez and Huerta show that under supersymmetry the Yang-Mills action changes by

$$\delta S = \int < \chi, (\epsilon . \chi) \chi >$$

  but this vanishes in D=3,4,6,10 by virtue of $\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$ $\mathbb{O}$ being *alternative*. See John's lecture.

5.1 Gravity Magic Pyramid

Tensoring left and right yields:

$D = 3$ A magic 4×4 square $\mathbb{RR}$, $\mathbb{CR}$, $\mathbb{CC}$, $\mathbb{HR}$, $\mathbb{HC}$, $\mathbb{HH}$, $\mathbb{OR}$, $\mathbb{OC}$, $\mathbb{OH}$, $\mathbb{OO}$ description of supergravities with $\mathcal{N} = 2, 3, 4, 5, 6, 8, 9, 10, 12, 16$.

$D = 4$ A 3×3 square $\mathbb{RR}$, $\mathbb{CR}$, $\mathbb{CC}$, $\mathbb{HR}$, $\mathbb{HC}$, $\mathbb{HH}$ description of supergravities with $\mathcal{N} = 2, 3, 4, 5, 6, 8$.

$D = 6$ A 2×2 square $\mathbb{RR}$, $\mathbb{CR}$, $\mathbb{CC}$ description of supergravities with $\mathcal{N} = 2, 3, 4$.

$D = 10$ A 1×1 square $\mathbb{RR}$ description of supergravities with $\mathcal{N} = 2$.

- Together these form *The Magic Pyramid*.
- The corresponding U-duality groups are given by a new algebraic structure, the magic pyramid formula, which may be regarded as being defined over three division algebras, one for spacetime and each of the left/right Yang-Mills multiplets.

5.2 Spacetime Fields in $D = n + 2$

The division algebras can be used to describe field theory in Minkowski space using the Lie algebra isomorphism [Sudbery: 1984]

$$\mathfrak{so}(1, 1 + n) \sim \mathfrak{sl}(2, \mathbb{A}).$$

- Spacetime spinors: $\mathbb{A}$-valued doublets.
- Spacetime vectors: $\mathbb{A}$-valued 2×2 Hermitian matrices.
- Connected to conventional notation via generalised $\mathbb{A}$-valued Paulis

Field Symbol	Representation	Rep. Symbol	Algebra
$\Psi_{\mathbb{A}}$	Spinor	S_+	$\mathfrak{so}(1, \dim \mathbb{A} + 1)$
$\mathcal{X}_{\mathbb{A}}$	Conjugate Spinor	S_-	$\mathfrak{so}(1, \dim \mathbb{A} + 1)$
$A_{\mathbb{A}}$	Vector	V	$\mathfrak{so}(1, \dim \mathbb{A} + 1)$
$\psi_{\mathbb{A}}$	Spinor	s	$\mathfrak{so}(\dim \mathbb{A})$
$\chi_{\mathbb{A}}$	Conjugate Spinor	c	$\mathfrak{so}(\dim \mathbb{A})$
$a_{\mathbb{A}}$	Vector	v	$\mathfrak{so}(\dim \mathbb{A})$

Table: A summary of the fields and notation used in $D = n + 2$

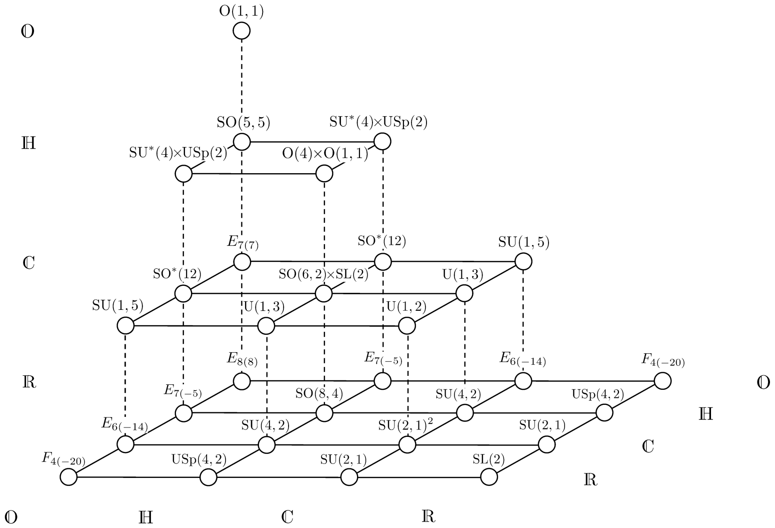
5.3 Super Yang-Mills in $D = \dim \mathbb{A} + 2 = 3, 4, 6, 10$

$\mathcal{N}$-extended super Yang-Mills can be formulated over the division algebras in $D = \dim \mathbb{A} + 2 = 3, 4, 6, 10$
[Kugo-Townsend: 1983, Evans: 1987, Baez-Huerta: 2009, MJD et al: 2013]

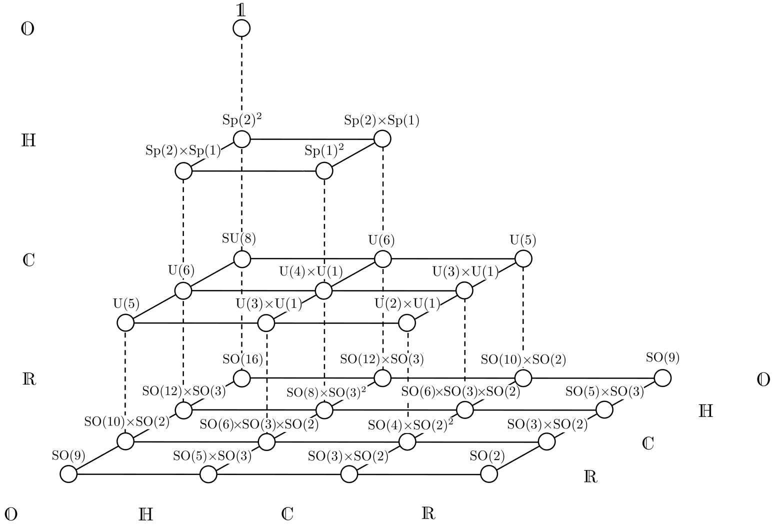
$D/\mathcal{N}$	1	2	4	8
10	$\mathrm{SO}(8)_{ST}$			
6	$\mathrm{SO}(4)_{ST} \times \mathrm{Sp}(1)_R$	$\mathrm{SO}(4)_{ST} \times \big(\mathrm{Sp}(1) \times \mathrm{Sp}(1)\big)_R$		
4	$\mathrm{SO}(2)_{ST} \times \mathrm{U}(1)_R$	$\mathrm{SO}(2)_{ST} \times \big(\mathrm{SU}(2) \times \mathrm{U}(1)^2\big)_R$	$\mathrm{SO}(2)_{ST} \times \mathrm{SU}(4)_R$	
3	1	$\mathrm{SO}(2)_R$	$\mathrm{SO}(4)_R$	$\mathrm{SO}(8)_R$

Table: Space-time Little and R-symmetry groups are related to the triality algebras of $\mathbb{A}$

5.4. Magic Pyramid: G symmetries



5.5 Magic Pyramid: H symmetries



5.6 Summary Gravity: Conformal Magic Pyramid

- We also construct a *conformal* magic pyramid by tensoring conformal supermultiplets in D = 3, 4, 6.

- Walls of pyramid now given by magic square too, except for the missing entry in D = 10

- Suggestive of an exotic theory with G/H duality structure $F_{4(4)}/Sp(3) \times Sp(1)$.

5.7 Conformal Magic Pyramid: G symmetries