Outline.

- **Introduction.**
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  - Some basic facts.
  - The case $n$ even.
    - Constructions of maximal partial transversals.
    - Constructions of transversals.
  - The case $n$ odd.
    - Results so far.
Latin squares.

Definition

A latin square of side $n$ is an $n \times n$ matrix in which each symbol from an $n$-element set appears exactly once in each row and each column.
Latin squares.

**Definition**

A *latin square* of side $n$ is an $n \times n$ matrix in which each symbol from an $n$-element set appears exactly once in each row and each column.

**Example**

A latin square of order 9.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 \\
3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 \\
4 & 1 & 2 & 3 & 8 & 9 & 5 & 6 & 7 \\
5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 1 \\
6 & 7 & 8 & 9 & 1 & 5 & 2 & 3 & 4 \\
7 & 8 & 9 & 5 & 4 & 1 & 6 & 2 & 3 \\
8 & 9 & 5 & 6 & 3 & 4 & 1 & 7 & 2 \\
9 & 5 & 6 & 7 & 2 & 3 & 4 & 1 & 8 \\
\end{pmatrix}
\]
Partial transversals.

Definition

A partial transversal of length $m$ in a latin square $L$ is a set $T$ of $m$ cells,

- at most one from each row,
- at most one for each column,
- no symbol of $L$ appearing more than once in $T$. 
Partial transversals.

**Definition**

A **partial transversal** of length $m$ in a latin square $L$ is a set $T$ of $m$ cells,
- at most one from each row,
- at most one for each column,
- no symbol of $L$ appearing more than once in $T$.

**Definition**

A **partial transversal** is **maximal** if it cannot be extended to a partial transversal of greater length.
A partial transversal of length 5 in a latin square of side 9.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 \\
3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 \\
4 & 1 & 2 & 3 & 8 & 9 & 5 & 6 & 7 \\
5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 1 \\
6 & 7 & 8 & 9 & 1 & 5 & 2 & 3 & 4 \\
7 & 8 & 9 & 5 & 4 & 1 & 6 & 2 & 3 \\
8 & 9 & 5 & 6 & 3 & 4 & 1 & 7 & 2 \\
9 & 5 & 6 & 7 & 2 & 3 & 4 & 1 & 8 \\
\end{bmatrix}
\]

This partial transversal is maximal.
A partial transversal of length 5 in a latin square of side 9.

\[
\begin{pmatrix}
\cdot \cdot \cdot \cdot & 5 & \cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot & 7 & \cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot & 9 & \cdot \cdot \cdot \\
5 & 6 & 7 & 8 & \cdot \cdot \cdot \cdot \\
6 & 7 & 8 & 9 & \cdot \cdot \cdot \cdot \\
7 & 8 & 9 & 5 & \cdot \cdot \cdot \cdot \\
8 & 9 & 5 & 6 & \cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot \cdot \cdot \cdot & 8 & \cdot \cdot \cdot \cdot \\
\end{pmatrix}
\]

This partial transversal is maximal.
A partial transversal of length 7 in a latin square of side 9.

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & \color{blue}5 & 6 & 7 & 8 & 9 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 \\
3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 \\
4 & 1 & 2 & 3 & 8 & 9 & 5 & 6 & 7 \\
5 & 6 & 7 & 8 & 9 & \color{blue}2 & \color{blue}3 & \color{blue}4 & 1 \\
6 & 7 & 8 & 9 & 1 & 5 & 2 & 3 & 4 \\
7 & 8 & 9 & 5 & 4 & 1 & 6 & 2 & 3 \\
8 & 9 & 5 & 6 & 3 & 4 & 1 & 7 & 2 \\
9 & 5 & 6 & 7 & 2 & 3 & 4 & 1 & 8 \\
\end{array}
\]
A partial transversal of length 7 in a latin square of side 9.

This partial transversal is maximal.
Notes on partial transversals.

$L$ a latin square of side $n$.
- A partial transversal of length $n$ is a transversal.
Notes on partial transversals.

$L$ a latin square of side $n$.

- A partial transversal of length $n$ is a **transversal**.
- A partial transversal of length $n-1$ is a **near transversal**.
Notes on partial transversals.

Let $L$ be a latin square of side $n$.

- A partial transversal of length $n$ is a transversal.
- A partial transversal of length $n-1$ is a near transversal.

**Theorem**

If a latin square of side $n$ has a maximal partial transversal of length $m$, then

$$\left\lfloor \frac{n}{2} \right\rfloor \leq m \leq n.$$
$L$ a latin square of side $n$.

- A partial transversal of length $n$ is a **transversal**.
- A partial transversal of length $n - 1$ is a **near transversal**.

**Theorem**

*If a latin square of side $n$ has a maximal partial transversal of length $m$, then*

$$\left\lfloor \frac{n}{2} \right\rfloor \leq m \leq n.$$  

**Example**

Our latin square of side 9 has maximal partial transversals of lengths 5, 6, 7, 8, and 9, i.e., all allowed lengths.
Notation.

$M_n$ will denote the Cayley table of $\mathbb{Z}_n$, i.e., the latin square of side $n$ and $ij$th entry

$$i + j \mod n,$$

$i, j = 0, \ldots, n - 1$. 
Bicyclic latin squares.

A latin square $L_{A,d_1,...,d_n}$ of side $2n + 1$ is **bicyclic** if its symbol set is

$$\mathbb{Z}_n \cup \mathbb{Z}_{n+1} = \{0, 1, \ldots, n-1\} \cup \{0, 1, \ldots, n\},$$

and

$$L_{A,d_1,...,d_n} = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

where
Bicyclic latin squares.

A latin square $L_{A,d_1,...,d_n}$ of side $2n + 1$ is bicyclic if its symbol set is

$$\mathbb{Z}_n \cup \mathbb{Z}_{n+1} = \{0, 1, \ldots, n - 1\} \cup \{0, 1, \ldots, n\},$$

and

$$L_{A,d_1,...,d_n} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

- $A$ is isotopic to $M_n$,
A latin square \( L_{A,d_1,\ldots,d_n} \) of side \( 2n + 1 \) is \textit{bicyclic} if its symbol set is
\[
\mathbb{Z}_n \cup \mathbb{Z}_{n+1} = \{0, 1, \ldots, n - 1\} \cup \{0, 1, \ldots, n\},
\]
and
\[
L_{A,d_1,\ldots,d_n} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]
where
\begin{itemize}
  \item \( A \) is isotopic to \( M_n \),
  \item \( B \) is \( M_{n+1} \) with the last row removed,
\end{itemize}
A latin square $L_{A,d_1,...,d_n}$ of side $2n + 1$ is **bicyclic** if its symbol set is

$$\mathbb{Z}_n \cup \mathbb{Z}_{n+1} = \{0,1,\ldots,n-1\} \cup \{0,1,\ldots,n\},$$

and

$$L_{A,d_1,...,d_n} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

- $A$ is isotopic to $M_n$,
- $B$ is $M_{n+1}$ with the last row removed,
- $C$ is $M_{n+1}$ with the last column removed, and
- $D$ has the last row of $B$/last column of $C$ on the main diagonal, and

$d_j - i \in \mathbb{Z}_n$, $i \neq j$. 

$A$ is isotopic to $M_n$, 

$B$ is $M_{n+1}$ with the last row removed,

$C$ is $M_{n+1}$ with the last column removed, and
Bicyclic latin squares.

A latin square $L_{A,d_1,...,d_n}$ of side $2n+1$ is **bicyclic** if its symbol set is

$$\mathbb{Z}_n \cup \mathbb{Z}_{n+1} = \{0, 1, \ldots, n-1\} \cup \{0, 1, \ldots, n\},$$

and

$$L_{A,d_1,...,d_n} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

- $A$ is isotopic to $M_n$,
- $B$ is $M_{n+1}$ with the last row removed,
- $C$ is $M_{n+1}$ with the last column removed, and
- $D$ has the last row of $B$/last column of $C$ on the main diagonal, and $ij$th entry $d_{j-i} \in \mathbb{Z}_n, i \neq j$. 
Bicyclic latin squares.

A bicyclic latin square of side 9.

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 0 \\
2 & 3 & 4 & 0 & 1 \\
3 & 4 & 0 & 1 & 2 \\
\end{pmatrix}

This is $L_{M_{4,1,2,3,0}}$. 
Some basic facts.

Lemma

If $n$ is even and a bicyclic latin square of side $2n + 1$ has a maximal partial transversal of length $m$, then

$$n + 1 \leq m \leq 2n + 1.$$
Some basic facts.

**Lemma**

*If* $n$ *is even and a bicyclic latin square of side* $2n + 1$ *has a maximal partial transversal of length* $m$, *then*

$$n + 1 \leq m \leq 2n + 1.$$  

**Lemma**

*If* $n$ *is even, then* $M_n$

- *has no transversals, and*
- *any cell can be the missing cell of a near transversal.*
Some basic facts.

Lemma

If $n$ is even and a bicyclic latin square of side $2n + 1$ has a maximal partial transversal of length $m$, then

$$n + 1 \leq m \leq 2n + 1.$$ 

Lemma

If $n$ is even, then $M_n$

- has no transversals, and
- any cell can be the missing cell of a near transversal.

Lemma

If $n$ is odd, then $M_n$

- has transversals, and
- any near transversal can be extended to a transversal.
The case $n$ even. Constructions of maximal partial transversals.

**Lemma**

If $n$ is even, then any bicyclic latin square of side $2n + 1$ has a maximal partial transversal of length $n + 1$. 
The case $n$ even. Constructions of maximal partial transversals.

**Lemma**

If $n$ is even, then any bicyclic latin square of side $2n + 1$ has a maximal partial transversal of length $n + 1$.

**The construction.**

For

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

pick a

*“transversal” of $B$, and*
The case $n$ even. Constructions of maximal partial transversals.

**Lemma**

If $n$ is even, then any bicyclic latin square of side $2n+1$ has a maximal partial transversal of length $n+1$.

**The construction.**

For $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, pick a

- “transversal” of $B$, and
- one entry on the main diagonal of $D$.  

The case \( n \) even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 1 & 3 & 0 & 1 & 1 \\
3 & 4 & 0 & 1 & 2 & 2 & 3 & 0 & 2 \\
4 & 0 & 1 & 2 & 3 & 1 & 2 & 3 & 0 \\
\end{array}
\]

The black entries form a maximal partial transversal of length 5.
The case $n$ even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

$$
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
\end{array}
\begin{array}{cccc}
& & & \\
& 2 & & \\
& & 4 & \\
& 1 & & \\
\end{array}
\begin{array}{cccc}
& & & \\
& 1 & 2 & 3 \\
& 0 & 1 & 2 \\
& 3 & 0 & 1 \\
& 2 & 3 & 0 \\
& 1 & 2 & 3 \\
\end{array}
\begin{array}{cccc}
& & & \\
& & & 3 \\
& & & 1 \\
\end{array}
$$

The black entries form a maximal partial transversal of length 5.
The case $n$ even. Constructions of maximal partial transversals.

**Lemma**

If $n$ is even, then any bicyclic latin square of side $2n + 1$ has a maximal partial transversal of length $2n$. 
The case $n$ even. Constructions of maximal partial transversals.

**Lemma**

If $n$ is even, then any bicyclic latin square of side $2n + 1$ has a maximal partial transversal of length $2n$.

**The construction.**

For 

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

pick a

- near transversal of $A$, and
The case \( n \) even. Constructions of maximal partial transversals.

**Lemma**

If \( n \) is even, then any bicyclic latin square of side \( 2n + 1 \) has a maximal partial transversal of length \( 2n \).

**The construction.**

For

\[
L = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]

pick a
- near transversal of \( A \), and
- the main diagonal of \( D \).
The case $n$ even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 0 \\
3 & 4 & 0 & 1 \\
4 & 0 & 1 & 2 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 0 \\
3 & 4 & 0 & 1 \\
\hline
0 & 1 & 2 & 3 \\
4 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 \\
3 & 0 & 1 & 1 \\
2 & 3 & 0 & 2 \\
\end{array}
\]

The entries shown, less one 1 form a maximal partial transversal of length 8.
The case $n$ even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

$$\begin{pmatrix}
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\end{pmatrix}$$

The entries shown, less one $1$, form a maximal partial transversal of length 8.
The case \( n \) even. Constructions of maximal partial transversals.

**Lemma**

If \( n \) is even and

\[ 1 \leq k < \frac{n}{2}, \]

then \( L_{M_n,1,...,n-1,0} \) has a maximal partial transversal of length \( n + 2k + 1 \).
The case $n$ even. Constructions of maximal partial transversals.

The construction.

First pick the black entries shown below.

\[
\begin{pmatrix}
0 & \cdots & 2(k - 1) \\
2k & \cdots & 2(n - 1) \\
& \cdots & \\
& & \ \ \ &= 2n
\end{pmatrix}
\]

Then choose blue entries.
The case $n$ even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3 \\
\end{bmatrix}
\]

The entries shown form a partial transversal of length 5, which can be extended. The red and black entries form a maximal partial transversal of length 7.
The case $n$ even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

\[
\begin{array}{cccccc}
0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 2 \\
\cdot & \cdot & \cdot & \cdot & 2 & 3 \\
\cdot & \cdot & \cdot & 4 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & 3 \\
\end{array}
\]

The entries shown form a partial transversal of length 5, which can be extended. The red and black entries form a maximal partial transversal of length 7.
The case $n$ even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

\[
\begin{array}{ccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\end{array}
\]

The entries shown form a partial transversal of length 5, which can be extended. The red and black entries form a maximal partial transversal of length 7.
The case $n$ even. Constructions of maximal partial transversals.

**Lemma**

If $n$ is even and

$$1 \leq k < \frac{n}{2},$$

then $L_{M_{n,1,\ldots,n-1,0}}$ has a maximal partial transversal of length $n + 2k$. 
The case $n$ even. Constructions of maximal partial transversals.

The construction.

First pick the black entries shown below.

\[
\begin{pmatrix}
0 \\
\vdots \\
2(n-1)
\end{pmatrix}
\begin{pmatrix}
2k & 2n \\
\end{pmatrix}
\begin{pmatrix}
0 \\
\ldots \\
2(k-1)
\end{pmatrix}
\]

Then choose blue entries.
The case $n$ even. Constructions of transversals.

A question.

For

$$L_{A,d_1,...,d_n} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

a bicyclic latin square of side $2n + 1$, $n$ even, when can a near transversal of $A$ be extended to a transversal of $L_{A,d_1,...,d_n}$?
The case $n$ even. Constructions of transversals.

A question.

For

$$L_{A,d_1,\ldots,d_n} = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

a bicyclic latin square of side $2n+1$, $n$ even, when can a near transversal of $A$ be extended to a transversal of $L_{A,d_1,\ldots,d_n}$?

A possible extension.

Let $T$ be a near transversal of $A$.

- Missing cell $(i,j)$ with entry $a$. 
The case \( n \) even. Constructions of transversals.

A question.

For

\[
L_{A,d_1,...,d_n} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]

a bicyclic latin square of side \( 2n + 1 \), \( n \) even, when can a near transversal of \( A \) be extended to a transversal of \( L_{A,d_1,...,d_n} \)?

A possible extension.

Let \( T \) be a near transversal of \( A \).

- Missing cell \((i,j)\) with entry \(a\).
- Missing symbol \(-a \in \mathbb{Z}_n\).
The case \( n \) even. Constructions of transversals.

A possible extension.

- Pick cell \((i, t)\) in \( B \): entry \( y = i + t \in \mathbb{Z}_{n+1} \).
The case $n$ even. Constructions of transversals.

A possible extension.

- Pick cell $(i, t)$ in $B$: entry $y = i + t \in \mathbb{Z}_{n+1}$.
- Pick cell $(s, j)$ in $C$: entry $x = s + j \in \mathbb{Z}_{n+1}$. 
The case $n$ even. Constructions of transversals.

A possible extension.

$$
\begin{pmatrix}
    j & t & s \\
    \vdots & \vdots & \vdots \\
    i & a & \ldots & y & \ldots & \ldots \\
    \vdots & \vdots & \vdots \\
    t & \ldots & \ldots & x & \ldots & ? \\
    \vdots & \vdots & \vdots \\
    s & \ldots & x & \ldots & \ldots & y \\
    \vdots & \vdots \\
\end{pmatrix}
$$
The case $n$ even. Constructions of transversals.

A possible extension.

We find that $T$ can be extended to a transversal if and only if

\begin{equation}
i + j + 2 \equiv 0 \pmod{n+1},\end{equation}

i.e., the missing cell of $T$ is on the antidiagonal of $A$, and

$? = -a \in \mathbb{Z}_n$. 
The case $n$ even. Constructions of transversals.

A possible extension.

We find that $T$ can be extended to a transversal if and only if

- $i + j + 2 \equiv 0 \pmod{n+1}$, i.e., the missing cell of $T$ is on the antidiagonal of $A$, and
A possible extension.

We find that $T$ can be extended to a transversal if and only if

1. $i + j + 2 \equiv 0 \pmod{n + 1}$, i.e., the missing cell of $T$ is on the antidiagonal of $A$, and
2. the “?” is $-a \in \mathbb{Z}_n$. 
The case \( n \) even. Constructions of transversals.

A bicyclic latin square of side 9.

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & \\
2 & 3 & 0 & 1 & \\
3 & 0 & 1 & 2 & \\
\hline
0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 0 \\
2 & 3 & 4 & 0 & 1 \\
3 & 4 & 0 & 1 & 2 \\
4 & 0 & 1 & 2 & \\
\end{array}
\]

The entries shown form a transversal.
The case \( n \) even. Constructions of transversals.

A bicyclic latin square of side 9.

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
\hline
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 1 & 2 & 3 & 0 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 1 \\
4 & 0 & 1 & 2 & 3 & 0 & 1 & 2 \\
\end{array}
\]

The entries shown form a transversal.
The case $n$ even. Constructions of transversals.

A bicyclic latin square of side 9.

\[
\begin{bmatrix}
 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\end{bmatrix}
\]

The entries shown form a transversal.
The case $n$ even. Constructions of transversals.

A bicyclic latin square of side 9.

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3 \\
\end{pmatrix}
\]

The entries shown form a transversal.
The case $n$ even. Constructions of transversals.

A bicyclic latin square of side 9.

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 0 \\
2 & 3 & 4 & 0 & 1 \\
3 & 4 & 0 & 1 & 2 \\
\end{pmatrix}
\]

The entries shown form a transversal.
The case $n$ even. Constructions of transversals.

A bicyclic latin square of side 9.

$$
\begin{pmatrix}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\end{pmatrix}
$$

The entries shown form a transversal.
The case $n$ even. Constructions of transversals.

A bicyclic latin square of side 9.

\[
\begin{array}{cccccccc}
& & & & & & & 4 \\
& & & & & & & \\
& & & & & & 3 & \\
2 & & & & & & & \\
& & & & & & 0 & \\
& & & & & & & \\
& & & & 3 & & & \\
& & & & & & 0 & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
1 & & & & & & & \\
& & & & & & & \\
& & & & & & & \\
1 & & & & & & & \\
\end{array}
\]

The entries shown form a transversal.
The case $n$ even. The conclusion.

**Theorem**

*If $n$ is even, then there exists a bicyclic latin square of side $2n + 1$ that has maximal partial transversals of all allowed lengths.*
The case $n$ odd. Results so far.

**Theorem**

*If $n$ is odd, then there exists a bicyclic latin square of side $2n + 1$ that has maximal partial transversals of all allowed lengths except possibly $n + 1$ and $2n$.*