An Overview of Rhotrix Quasigroups & Rhotrix Loops[1] [2]

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Introduction

In 1998, Atanassov and Shannon discussed mathematical arrays that are in some way, between two-dimensional vectors and (2×2)-dimensional matrices in their paper denoted matrix-tertions and noitrets. Ajibade(2003) introduced objects which are in some ways, between (2×2)-dimensional and (3×3) dimensional matrices. Such an object is called a rhotrix in Ajibade (2003), and went further to define a real rhotrix as follows:

A Rhotrix

Definition

A rhotrix is a rhomboid array of numbers given as:

$$R = \{\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in \mathfrak{R}\}$$

where $c = h(R)$ is called the heart of any rhotrix R and $\mathfrak{R}$ is the set of real numbers. R is the set of all 3-dimensional rhotrices.

However, the paper observed that an extension of this set was possible in various ways, and also noted that the name rhotrix was a result of the rhomboid nature. Several authors have worked on the algebra, analysis and applications a rhotrix into different field of sciences-see Mohammed(2007), Aminu(2010) and others.

The algebra and analysis of rhotrices are presented in Ajibade(2003). Thus, addition and multiplication of two heart-based rhotrices are defined below: Let

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \text{and } Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

then

$$R + Q = \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array} \right\rangle$$

and

$$R \circ Q = \left\langle \begin{array}{ccc} & ah(Q) + fh(R) & \\ bh(Q) + gh(R) & h(R)h(Q) & dh(Q) + jh(R) \\ & eh(Q) + kh(R) & \end{array} \right\rangle$$

A generalization of this hearty multiplication is given in Mohammed et al (2011). A far-reaching observation was made in Ajibade(2003) that multiplication on rhotrices can be defined in many ways. The following year, Sani(2004) introduced an alternative method of rhotrix multiplication.

A row-column multiplication of heart-based rhotrices was proposed by Sani(2004) as:

$$R \circ Q = \left\langle \begin{array}{ccc} & af + dg & \\ bf + eg & h(R)h(Q) & aj + dk \\ & bj + ek & \end{array} \right\rangle$$

A generalization of this row-column multiplication was also later given by Sani(2007) as:

$$R_n \circ Q_n = \langle a_{ij}, c_{ij} \rangle \circ \langle b_{ij}, d_{lk} \rangle = \left\langle \sum_{i,j=1}^{t} (a_{ij} b_{ij}), \sum_{l,k=1}^{t-1} (c_{lk} d_{lk}) \right\rangle, t = (n+1)/2.$$

where R_n and Q_n are n-dimensional rhotrices (with n rows and n columns). This new method was not commutative unlike the former. These two definitions set a bearing for researches in rhotrix theory.

An Example of a Higher Rhotrix

(i) A *hl*-rhotrix of dimension four (R_5) is given by:

$$R_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle$$

Then its corresponding coupled matrix will be presented below:

(ii)

$$R_5^c = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} \\ & c_{11} & & c_{12} & \\ a_{21} & & a_{22} & & a_{23} \\ & c_{21} & & c_{22} & \\ a_{31} & & a_{32} & & a_{33} \end{bmatrix}$$

Some Areas of Application

(1) Solving $n \times n$ and $(n-1) \times (n-1)$ system of linear equations
(2) Application into graph, groups and semigroup theories
(3) In Computer science and Statistics
(4) Coding Theory and Cryptography

Motivation

That rhotrix multiplication can be be defined in many ways is a motivation for this work, Ajbade(2003) And that the only two rhotrix multiplication methods hitherto are both associative is another motivation. Several works on group of matrices, such as the work of Smith(2006) on left quasigroups, and Johnson and Vojtechovsky(2005) on right division in groups of matrices were all a motivation.

Several papers have characterized groups using the operation of right division $x \cdot y^{-1}$ instead of the multiplication $x \cdot y$. Johnson and Vojtechovskys said however that it was not clear how much is gained within group theory per see by such change in perspective. The aim of this paper however, is to show the significance of a similar change in rhotrix multiplication.

The Concept of Non-Associative Rhotrix Theory

This section presents the non-associative rhotrix multiplication. It begins with a particular example of a multiplication method that is non-associative and then prepares the stage for many definitions of rhotrix multiplications as observed earlier by Ajibade (2003) but with emphasis on the non-associative binary multiplications which could be commutative or non-commutative.

Definition

Let

$$\hat{R} = \{R : R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle, a, b, c, d, e \in \mathfrak{R}\} \tag{1}$$

be a set of all three dimensional rhotrices where $h(R) = c$ is called the heart of R.

Definition

Let $R, Q \in \hat{R}$, such that

$$R = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & h(R) & d_1 \\ & e_1 & \end{array} \right\rangle \mathit{and}\ Q = \left\langle \begin{array}{ccc} & a_2 & \\ b_2 & h(Q) & d_2 \\ & e_2 & \end{array} \right\rangle$$

then

$$R \odot Q = \left\langle \begin{array}{ccc} & \overline{a_1} a_2 & \\ \overline{b_1} b_2 & \overline{h(R)} h(Q) & \overline{d_1} d_2 \\ & \overline{e_1} e_2 & \end{array} \right\rangle$$

then, we call the operation above a left conjugate multiplication

Definition

Let $R, Q \in \hat{R}$, such that

$$R = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & h(R) & d_1 \\ & e_1 & \end{array} \right\rangle and\ Q = \left\langle \begin{array}{ccc} & a_2 & \\ b_2 & h(Q) & d_2 \\ & e_2 & \end{array} \right\rangle$$

then

$$R \odot Q = \left\langle \begin{array}{ccc} & a_1\overline{a_2} & \\ b_1\overline{b_2} & h(Q)\overline{h(Q)} & d_1\overline{d_2} \\ & e_1\overline{e_2} & \end{array} \right\rangle$$

then, we call the operation above a right conjugate multiplication

Remark

(i) *where* $\overline{a_1}$ *is the left conjugate of* a_2 *and* $\overline{a_2}$ *is the right conjugate of* a_1 *such that* $a\overline{a} = \overline{a}a = 1$ *and* $a1 = a = 1a$*. Then,* $(\hat{R}, \odot)$ *is a groupoid, called a rhotrix groupoid.* (ii) *Also,* $\overline{a} = a^{-1}$ *where 'a' is a real number, and the juxtaposition is a direct multiplication of real numbers.*

Behold, the left and right conjugate operations are equivalent to the left and right division operation. Therefore, we can redefine our left conjugate operation as:

Definition

Let $R, Q \in \hat{R}$, such that

$$R = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & h(R) & d_1 \\ & e_1 & \end{array} \right\rangle and\ Q = \left\langle \begin{array}{ccc} & a_2 & \\ b_2 & h(Q) & d_2 \\ & e_2 & \end{array} \right\rangle$$

then

$$R \odot Q = \left\langle \begin{array}{ccc} & a_1 \setminus a_2 & \\ b_1 \setminus b_2 & h(R) \setminus h(Q) & d_1 \setminus d_2 \\ & e_1 \setminus e_2 & \end{array} \right\rangle$$

such that

$$a \cdot (a \setminus b) = b \quad a \setminus (a \cdot b) = b \tag{2}$$

then, we call the operation above a left conjugate multiplication

Definition

Let $R, Q \in \hat{R}$*, such that*

$$R = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & h(R) & d_1 \\ & e_1 & \end{array} \right\rangle and\ Q = \left\langle \begin{array}{ccc} & a_2 & \\ b_2 & h(Q) & d_2 \\ & e_2 & \end{array} \right\rangle$$

then

$$R \odot Q = \left\langle \begin{array}{ccc} & a_1/a_2 & \\ b_1/b_2 & h(Q)/h(Q) & d_1/d_2 \\ & e_1/e_2 & \end{array} \right\rangle$$

such that

$$b = (b/a) \cdot a \quad b = (b \cdot a)/b \tag{3}$$

then, we call the operation above a right conjugate multiplication

Then we call the resultant rhotrix above a conjugate rhotrix (CR). An example of a CR is a rhotrix with rational entries. It is to be noted that a conjugate operation is strictly a left conjugate operation or a right conjugate operation.

Special Conjugate Rhotrix (SCR)

A conjugate rhotrix whose heart is invariant under conjugate operation is called an SCR.
Examples of SCR are

(1) The conjugate identity (Trivial)
(ii) Rhotrices with unit hearts
(iii) Rhotrices with equal heart
(iv) Rhotrix group of roots of unity

A left conjugate rhotrix quasigroup $(Q, \cdot)$ is a rhotrix Q equipped with a left conjugate operation such that for all a and b, there is a unique element c such that

$$a \cdot b = c \tag{4}$$

The right conjugate rhotrix quasigroup is defined analogously.

Equationally, a quasigroup $(Q, \cdot, /, \backslash)$ is a conjugate rhotrix quasigroup or simply a rhotrix quasigroup equipped with three binary operations of multiplication, right division $(/)$ and left division $(\backslash)$ satisfying the identities 2 and 3 respectively . These identities correspond respectively to the uniqueness of the solution 4.

Definition

2.6 Let $R, I \in \hat{R}$ and $\odot$ a conjugate operation. If

$$R \odot I = R = I \odot R$$

then, we called I a two sided conjugate identity, under $\odot$ as defined above. This implies that

$$I = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle$$

Definition

Let $R, X \in \hat{R}$ and $\odot$ a conjugate operation. If $R \odot X = I$ or $X \odot R = I$ then, we called X the conjugate inverse of R. That is

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \text{ implies that } R^{-1} = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

Then R is self-invertible or self-conjugate under $\odot$.

Remark

We can easily verify that conjugate operation on rhotrices as defined above is non-commutative and non-associative, except at trivial cases where the rhotrices are identical or when having the value of the heart repeated at every other point etc.

Lemma

Let A and B be distinct rhotrices of the same dimension in $\hat{R}$. Then, $A \odot X = B$ and $Y \odot A = B$ have unique solutions in $\hat{R}$ (unique solvability)

Definition

Let $(\hat{R}, \odot)$ be a rhotrix groupoid and let $I \in \hat{R}$. Then I is a left(right)identity rhotrix for $(\hat{R}, \odot)$ means that

$$C_L(I) : \hat{R} \mapsto \hat{R}(C_R(I) : \hat{R} \mapsto \hat{R})$$

is the identity conjugate operator on $\hat{R}$.

Definition

A rhotrix quasigroup with a left and right identity rhotrix is called a rhotrix loop

This means that a rhotrix groupoid $(\hat{R}, \odot)$ is a rhotrix loop if $(\hat{R}, \odot)$ is a rhotrix quasigroup that has a two-sided identity rhotrix. Thus, all rhotrix groups are rhotrix loops. But all rhotrix loops are not rhotrix groups. Those that are rhotrix groups are associative rhotrix loops. Therefore, rhotrix loops generalize rhotrix groups. It is worth noting that rhotrices as defined by Ajibade (2003) and Sani(2004) are associative rhotrix loops. These types of rhotrix loops are trivial rhotrix loops. Whereas, rhotrix loops defined by conjugate operation as considered in this work, are non-trivial rhotrix loops

Definition

A rhotrix groupoid $(\hat{R}, \odot)$ *is called a rhotrix quasigroup if the conjugate operators* $C_R(Q) : \hat{R} \mapsto \hat{R}$ *and* $C_L(Q) : \hat{R} \mapsto \hat{R}$ *are bijections. And if it possess in addition, an identity rhotrix, then* $(\hat{R}, \odot)$ *becomes a rhotrix loop. The order of* $\hat{R}$ *is its cardinality* $|\hat{R}|$.

Other Examples of Rhotrix Multiplications

This section confirms the observation in [2] that rhotrix multiplication can be defined in many ways. These multiplications are defined using Cayley tables. Depending on the definitions, rhotrix multiplications can be commutative, associative, non-commutative or non-associative, or even both. However, we are going to be concerned with non-associative rhotrix multiplications which could be commutative or non-commutative. Starting with the two multiplication methods already known in literature., we have the following

Example

Let

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle and\ Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

then

$$R \circ Q = \left\langle \begin{array}{ccc} & a \setminus h(Q) + h(R) \setminus f & \\ b \setminus h(Q) + h(R) \setminus g & h(R) \setminus h(Q) & d \setminus h(Q) + h(R) \setminus j \\ & e \setminus h(Q) + h(R) \setminus k & \end{array} \right\rangle$$

such that the identities (2) are satisfied.

Example

Let

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle and\ Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

then

$$R \circ Q = \left\langle \begin{array}{ccc} & a \setminus f + d \setminus g & \\ b \setminus f + e \setminus g & h(R) \setminus h(Q) & a \setminus j + d \setminus k \\ & b \setminus j + e \setminus k & \end{array} \right\rangle$$

such that the identities (2) are satisfied.

Remark

The two examples of rhotrix multiplication just presented are non-associative. They can be referred to as Left Division Multiplication (LDM) of the Ajibade and Sani multiplication methods respectively. The Right Division Multiplication (RDM) can be defined Analogously.

Example

Let $\hat{R} = \{I, P, Q\}$ *be a set of arbitrary rhotrices of the same dimension, and a binary multiplication* $(\cdot)$ *defined as*

$\cdot$	I	P	Q
I	I	P	Q
P	P	Q	I
Q	Q	I	P

Table : Associative and Commutative Rhotrix Loop

Then, $(\hat{R}, \cdot)$ is an associative and commutative rhotrix loop. A trivial loop.

Example

Let $\hat{R} = \{I, P, Q, R\}$ *be a finite set of arbitrary rhotrices of the same dimension. Define multiplication* $(\circ)$ *as*

$\circ$	I	P	Q	R
I	I	P	Q	R
P	P	I	R	Q
Q	Q	R	I	P
R	R	Q	P	I

Table : Associative and Commutative Rhotrix Loop

Then, $(\hat{R}, \circ)$ is also an associative and commutative rhotrix loop. This is also trivial.

Example

Let $\hat{R} = \{I, P, Q, R, S\}$ be a finite set of arbitrary rhotrices of the same dimension and $\odot$ be given by the table below:

$\odot$	I	P	Q	R	S
I	I	P	Q	R	S
P	P	I	R	S	Q
Q	Q	S	I	P	R
R	R	Q	S	I	P
S	S	R	P	Q	I

Table : Non-Associative and Non-Commutative Rhotrix Loop

Then, $(\hat{R}, \odot)$ is a non-trivial rhotrix loop.

Example

Let $\hat{R} = \{I, P, Q, R, S\}$ be a finite set of arbitrary rhotrices of the same dimension and $\odot$ be given by the table below:

$\odot$	I	P	Q	R	S
I	I	P	Q	R	S
P	P	I	R	S	Q
Q	Q	S	I	P	R
R	R	Q	S	I	P
S	S	R	P	Q	I

Table : Non-Associative and Non-Commutative Rhotrix Loop

Then, $(\hat{R}, \odot)$ is a non-trivial rhotrix loop.

Example

Let $\hat{R} = \{I, P, Q, R, S\}$ *be a finite set of arbitrary rhotrices of the same dimension and* $(\bullet)$ *be given by the table below:*

$\bullet$	I	P	Q	R	S
I	I	P	Q	R	S
P	P	Q	R	S	I
Q	Q	R	S	I	P
R	R	S	I	P	Q
S	S	I	P	Q	R

Table : A Commutative Rhotrix Group

Then, $(\hat{R}, \bullet)$ is a trivial rhotrix loop.

Example

Let $\hat{R} = \{I, P, Q, R, S, T, U, V\}$ *be a finite set of arbitrary rhotrices of the same dimension and let* $\odot$ *be defined by the table below:*

$\odot$	I	P	Q	R	S	T	U	V
I	I	P	Q	R	S	T	U	V
P	P	I	R	Q	T	S	V	U
Q	Q	R	I	P	V	U	S	T
R	R	Q	P	I	U	V	T	S
S	S	T	U	V	I	P	Q	R
T	T	S	V	U	P	I	R	Q
U	U	V	S	T	R	Q	I	P
V	V	U	T	S	Q	R	P	I

Table : Non-Associative and Non-Commutative Rhotrix Loop

Definition

Let $(\hat{R}, \odot)$ be a rhotrix groupoid and let Q be any fixed rhotrix in $\hat{R}$. Then, C_R is a right conjugate operator if

$$PC_R(Q) = P \odot Q$$

and a left conjugate operator if

$$PC_L(Q) = Q \odot P$$

for all $P \in \hat{R}$. It follows that $C_R(Q) : \hat{R} \mapsto \hat{R}$ and $C_L(Q) : \hat{R} \mapsto \hat{R}$ for each $Q \in \hat{R}$.

Remark

Whenever, the operation is not a conjugate operation, the conjugate operators are simply the translation maps i.e $C_R(Q) = R_Q$ and $C_L(Q) = L_Q$ -see Pflugfelder(1990)

Definition

A rhotrix groupoid $(\hat{R}, \odot)$ *is commutative means that*

$$C_L(Q) = C_R(Q)$$

for all $Q \in \hat{R}$

Definition

A rhotrix groupoid $(\hat{R}, \odot)$ *is associative if the conjugate operator*

$$C_R(Q \odot P) = C_R(Q)C_R(P)$$

for all $Q, P \in \hat{R}$

Remark

These definitions above are helpful in determining whether or not a rhotrix multiplication defined usually by a Cayley tables are commutative, associative or otherwise.

Lemma

The heart of a conjugate rhotrix corresponds to the center of a rhotrix quasigroup(loop)

Theorem

Let $(\hat{R}, \odot)$ be a rhotrix groupoid. the following are equivalent:

(i) $(\hat{R}, \odot)$ *is a rhotrix quasigroup.*

(ii) $C_R(Q) : \hat{R} \mapsto \hat{R}$ *and* $C_L(Q) : \hat{R} \mapsto \hat{R}$ *are injective for all* $Q \in \hat{R}$.

(iii) $C_R(Q) : \hat{R} \mapsto \hat{R}$ *and* $C_L(Q) : \hat{R} \mapsto \hat{R}$ *are surjective for all* $Q \in \hat{R}$.

(iv) *The left and right cancellation laws hold for* $(\hat{R}, \odot)$.

Corollary

Let $(\hat{R}, \odot)$ be a quasigroup. Then, the following hold:

(i) *For $R, Q, P \in \hat{R}$, $R \odot Q = R \odot P$ implies $P = Q$ (left cancellation law)*

(ii) *For $R, Q, P \in \hat{R}$, $Q \odot R = P \odot R$ implies $P = Q$ (right cancellation law)*

Theorem

The heart of an SCR commutes and associates

Proof

Let R and Q be two rhotrices of the same dimension.
Consider:

$$R \odot Q = h(R)\overline{h(Q)} = \overline{h(R)}h(Q) = Q \odot R$$

Then, the heart commutes.
Next, we show associativity. Let R, Q and P be three rhotrices

$$(R \odot Q) \odot P = (h(R)\overline{h(Q)}) \odot P = (h(R)\overline{h(Q)})\overline{h(P)} = h(R)(\overline{h(Q)h(P)}) = h$$

The heart associates.

Conclusion

This work opens up a large door of research to exploit the properties of rhotrices as binary systems. Though, rhotrices are geometric objects, but using algebra as a microscope, one is able to examine the scope of their properties. This is a reminiscence of the age long interplay between geometry and algebra. Therefore, there is need to investigate rhotrices through a geometric approach. This article examined the properties of the rhotrix through non-associative binary systems.

Many things in nature are not linear. Thus, assuming linearity on them limits the much we can know about them. For example, in a rhotrix loop, there may be some rhotrices or a rhotrix that may commute or associate with every other rhotrix in the loop. Such a rhotrix may exist at the heart of the rhotrix loop. It is interesting to find out such a rhotrix. These are areas for future work.

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