Four dimensional Euclidean gravity and octonions

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Outline

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   - 2) Linearized gravity from global coordinates
2. Four dimensional Euclidean gravity
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   - Octonion exponentiation?
   - Lattice numbers?

1) Dirac equation on split-octonions

Eight linear differential equations over $\mathbb{R}$

The field-free Dirac equation in Dirac representation (with all elegance from phycis removed):

$$\begin{pmatrix} -m+i\partial_0 & 0 & -i\partial_3 & -i\partial_1-\partial_2 \\ 0 & -m+i\partial_0 & -i\partial_1+\partial_2 & i\partial_3 \\ i\partial_3 & i\partial_1+\partial_2 & -m-i\partial_0 & 0 \\ i\partial_1-\partial_2 & -i\partial_3 & 0 & -m-i\partial_0 \end{pmatrix} \psi = 0,$$

over $\mathbb{C}^4$, with $\psi = \left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}},\ \ \psi_1^{\mathrm{r}} + i\psi_1^{\mathrm{i}},\ \ \psi_2^{\mathrm{r}} + i\psi_2^{\mathrm{i}},\ \ \psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right)\ldots$

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...written as split-octonion product over $\mathbb{R}^8$ ([1] eq 14):

$$\left(-m, \partial_0, 0, 0,\ 0, -\partial_3, \partial_2, -\partial_1\right)\left(\psi_0^{\mathrm{r}}, \psi_0^{\mathrm{i}}, \psi_1^{\mathrm{r}}, \psi_1^{\mathrm{i}},\ \psi_2^{\mathrm{r}}, -\psi_2^{\mathrm{i}}, -\psi_3^{\mathrm{r}}, -\psi_3^{\mathrm{i}}\right) = 0.$$

Same 8 equations over $\mathbb{R}$, different algebra $\rightarrow$ different gauges.

2) Linearized gravity from global coordinates

Minkowski spacetime + varying potential for any two point masses = linearized gravity

- SRT geometry (flat Minkowski, unaccelerated observer).
- Modify gravity between two point masses ([2] eq (58) (*)):

$$V = \frac{m}{|\vec{x}|} \quad \longrightarrow \quad V' = \frac{m}{|\vec{x}|}\left(1 + |\vec{v}|^2\right)$$

- SRT generalization of Newton gravity ([3] exer 7.3/box 7.1):

$$\Box h_{\mu\nu} = -8\pi M_{\mu\nu}$$

- Apply modified gravity for one point mass, bring into tensor form, generalize for arbitrary number of point masses.
- Results in linearized GRT.

Inconsequential or deeper context?

Two curious calculations ...

... Entertaining but inconsequential?

... The result from a deeper underlying context?

Four dimensional Euclidean gravity

Native spacetime geometries: Minkowski for EM, 4D Euclidean for gravity

Proposed context: "Naturally Aligned Equations" (NatAliE)

4D Euclidean and 4D Minkowskian spacetimes are the native geometries from the unaccelerated observer under the influence of gravity and electromagnetism, respectively ([2] prop 4).

Minkowski (SRT):

$$\begin{aligned} d\tau &= dt\sqrt{1-|\vec{v}|^2}, \\ L_{0,\parallel} &= \frac{L_\parallel}{\sqrt{1-|\vec{v}_\parallel|^2}}, \\ m_0 &= m\sqrt{1-|\vec{v}|^2}, \end{aligned}$$

4D Euclidean:

$$\begin{aligned} d\tau' &= dt\sqrt{1+|\vec{v}|^2}, \\ L'_0 &= \frac{L}{\sqrt{1+|\vec{v}|^2}}, \\ m'_0 &= m\sqrt{1+|\vec{v}|^2}. \end{aligned}$$

Four dimensional Euclidean gravity

Projecting 4D Euclidean gravity into Minkowski spacetime

Projecting Euclidean length $L \to L/\sqrt{1+|\vec{v}|^2}$ and mass $m \to m\sqrt{1+|\vec{v}|^2}$ yields [2] eq (58) (*):

$$V = \frac{m}{|\vec{x}|} \quad \longrightarrow \quad V = \frac{m}{|\vec{x}|}\left(1+|\vec{v}|^2\right).$$

(*) The 2007 paper [2] uses an additional factor $\gamma = 1/\sqrt{1-|\vec{v}|^2}$ which carries through the calculation and has to be explained as negligible at the end. However, the factor shouldn't be there from starters.

Consequence and caution

- If NatAliE foundational $\longrightarrow$ rework basic principles of physics: equivalent frames of reference, geometry of flat space, speed of light.
- Any large body, non-quantum model carries doubt of approximation since nature at its building blocks is quantum.
- EM length contraction $L_{0,\|} = L_\| / \sqrt{1 - |\vec{v}_\| |^2}$ goes in the direction of relative motion, whereas the $L \rightarrow L/\sqrt{1 + |\vec{v}|^2}$ goes along the respective directions between each two point masses.

Linearized GRT is nice, but to really support NatAliE there has to be a quantum theory that works without handwaving.

Dirac equation with electromagnetic field

Dirac equation with EM field: e^{ieA} gauge

$$\partial_\mu \quad \xrightarrow{ieA} \quad \partial_\mu + ieA^*_\mu$$

Identical gauge for $(-m, \partial_0, 0, 0,\ \ 0, -\partial_3, \partial_2, -\partial_1)$ on split-$\mathbb{O}$ to basis $\{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ requires complex-$\mathbb{O}$ to basis $\{1, i_1, \ldots i_7,\ \ i_0, \varepsilon_1, \ldots, \varepsilon_7\}$:

$$(0, \partial_0, 0, \ldots, 0, -\partial_3, \partial_2, -\partial_1) \xrightarrow{i_0 eA}$$
$$(0, \partial_0, 0, \ldots, 0, -\partial_3, \partial_2, -\partial_1) + (\ldots, 0, eA_3, -eA_2, eA_1,\ \ 0, eA_0, 0, \ldots).$$

Gravielectric field in complex octonions

Rotate $\alpha \in \mathbb{R}$ between Minkowskian $\alpha = \frac{\pi}{2}$ and 4D Euclidean $\alpha = 0$ geometry:

$$
\begin{aligned}
\nabla^{\text{Gr,EM}}_{\text{Q1}} &:= \left(0, \partial_0, 0, 0, 0, 0, 0, 0, \quad 0, eA_0, 0, 0, 0, 0, 0, 0\right) \\
\nabla^{\text{Gr,EM}}_{\text{Q2}} &:= \left(0, 0, 0, 0, 0, \partial_3, -\partial_2, \partial_1, \quad 0, 0, 0, 0, 0, eA_3, -eA_2, eA_1\right) \\
\nabla^{\text{Gr,EM}} &:= \nabla^{\text{Gr,EM}}_{\text{Q1}} + \exp\left(i_0 \alpha\right) \nabla^{\text{Gr,EM}}_{\text{Q2}} \\
\psi^{\text{Gr,EM}}_{\text{Q1}} &:= \left(\psi_0^{\text{r}}, \psi_0^{\text{i}}, \psi_1^{\text{r}}, \psi_1^{\text{i}}, 0, 0, 0, 0, \quad 0, 0, 0, 0, 0, 0, 0, 0\right) \\
\psi^{\text{Gr,EM}}_{\text{Q2}} &:= \left(0, 0, 0, 0, -\psi_2^{\text{r}}, \psi_2^{\text{i}}, \psi_3^{\text{r}}, \psi_3^{\text{i}}, \quad 0, 0, 0, 0, 0, 0, 0, 0\right) \\
\psi^{\text{Gr,EM}} &:= \psi^{\text{Gr,EM}}_{\text{Q1}} + \exp\left(i_0 \alpha\right) \psi^{\text{Gr,EM}}_{\text{Q2}}
\end{aligned}
$$

Yields equation of motion with gravielectric field ([4] eq 23):

$$
\left(\nabla^{\text{Gr,EM}} - m\right) \psi^{\text{Gr,EM}} = 0
$$

Different from *ict*

Compared to Euclidean quantum gravity (field theory)

Comparison with textbook use of 4D Euclidean geometry in physics:

Euclidean quantum gravity:	**NatAliE with complex $\mathbb{O}$**:
Well-known, established field theory of gravitation	Equation of motion of a particle (i.e., not a field theory)
Maps between Minkowskian and Euclidean geometry through imaginary time, *ict* ("Wick rotation")	Smooth transition between Minkowskian and Euclidean geometry through real parameter α in the algebra (all physical parameters affected)

Where from here?
Tantalizing hints from $\mathbb{O}$

Calculate gravielectric effects, e.g.:

- elastic scattering for $\alpha = 0$ [5] or general α,
- vacuum polarization of a point charge as function of α,
- bound states.

Tantalizing octonion hints towards Standard Model:

- $su(2)$ Lie subalgebra,
- transformations between basis elements of $\mathbb{O}$ that leave one imaginary basis unchanged form the $\mathrm{SU}(3)$ Lie group.
- the $su(3)$ Lie algebra can readily be built in $\mathbb{O}$.

Try to generalize octonionic Dirac equation: more algebraic freedom with octonion structure. But how??

Octooctonions?

Octonions with octonion coefficients

- Enlarge both operator and wave function to allow for octonion coefficients [6], octooctonion algebra $\mathbb{O} \times \mathbb{O}$ built over $\mathbb{R}^{64}$:

$$\begin{aligned} \Psi^{64} &:= \left\{ \psi : \mathbb{R}^4 \to \mathbb{O} \times \mathbb{O} \right\} \\ \psi_j &\in \nabla_k \Psi^{64}, \qquad \nabla_k : \Psi^{64} \to \Psi^{64}, \\ j,k &= 1,2,\ldots, \end{aligned}$$

- The ∇_j and ψ_k mix somehow to form $(\nabla - m)\,\psi = 0$.
- Problem: $\mathbb{O} \times \mathbb{O}$ is not alternative or flexible, doesn't have a multiplicative norm or other real magnitude; probability becomes ill-defined:

$$\int \psi\,(\nabla \psi) \neq \int (\psi \nabla)\,\psi.$$

Octooctonions?
Other 64D in physics

Functions $f : \mathbb{R}^4 \to \mathbb{R}^{64}$ are not unheard of in fundamental physics:

- Christoffel symbols Γ^i_{jk} $(i, j, k = 0, 1, 2, 3)$ in conventional General Relativity,
- Past (Dixon [7]) and current (Furey [8]) reserach of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ with constraint: Physics happens in algebraic ideals.

Octonion exponentiation?

Find an operation on octonions that has properties from complex exponentiation

Rather than modeling Dirac equation from a product

$$(-m, \partial_0, 0, 0,\ \ 0, -\partial_3, \partial_2, -\partial_1)\left(\psi_0^{\mathrm{r}}, \psi_0^{\mathrm{i}}, \psi_1^{\mathrm{r}}, \psi_1^{\mathrm{i}},\ \ \psi_2^{\mathrm{r}}, -\psi_2^{\mathrm{i}}, -\psi_3^{\mathrm{r}}, -\psi_3^{\mathrm{i}}\right) = 0,$$

assume wave functions ψ as product of some octonion "exponentials" (to be found / defined),

$$\begin{aligned} \Psi^8 &:= \left\{\psi : \mathbb{R}^4 \to \mathbb{O}\right\}, \\ \psi &:= \prod_j \alpha_j^{\beta_j} \qquad \left(\alpha_j, \beta_j, \psi \in \Psi^8\right), \end{aligned}$$

and look for $\hat{D} : \Psi^8 \to \Psi^8$ to find solutions of the form:

$$\left(\hat{D} - \lambda\right)\psi = 0 \qquad (\lambda \in \mathbb{R}).$$

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Why?

Octonion exponentiation?

Suggestive structure in primitive complex subspaces with one parameter

In the complex subspaces we find solutions that mimic the Dirac equation with $1/x$ fields in one dimension [9 eq 3.1]:

$$\begin{aligned}
\psi(x) &:= i^{t_n x}\left(\prod_{j=1}^{n-1}|x-a_j|^{it_j}\right) \qquad (x, t_j, a_j \in \mathbb{R}) \\
&= \exp\left(i\frac{\pi}{2}t_n x\right)\prod_{j=1}^{n-1}\exp\left(it_j \ln|x-a_j|\right), \\
\hat{D} &:= -i\frac{\partial}{\partial x} - \sum_{j=1}^{n-1}\frac{t_j}{x-a_j}, \\
&\Rightarrow \left(\hat{D} - \frac{\pi}{2}t_n\right)\psi = 0.
\end{aligned}$$

$\rightarrow$ Algebraic motivation of $1/x$ fields?!? ...

Octonion exponentiation?

Putting constraints on such operation

What constraints to put on octonion exponentiation? Maybe...

- $\mathbb{C}$ subspaces to "behave like $\mathbb{C}$ exponentiation" (e.g. growth, power associative),
- Exponential of two imaginary basis elements $i_j, i_k \in \mathbb{O}$ $(j \neq k)$ to be $\mathbb{O}$ multiplication [10]:

$$i_j^{i_k} \stackrel{!}{=} \exp\left(\ln\left(i_j^{i_k}\right)\right) := \exp\left(\ln\left(i_j\right) i_k\right) = \exp\left(\frac{\pi}{2} i_j i_k\right) = i_j i_k \,.$$

- Since such exponentiation intersects with multiplication at the imaginary basis elements, there should be sufficient algebraic structure to recover the Dirac equation again.

Is this possible? Lots of problems to be solved.

Octonion exponentiation?

Try: Exponentiation defined from multiplication with a suitable logarithm:

$$a^b := (\ln a)\, b \qquad (a, b \in \mathbb{O})$$

1. Pass: Satisfies constraints.
2. Pass: Algebra is flexible $\Rightarrow$ measurements $\int \psi (\nabla \psi)$.
3. Fail: No conceptual gain: Ad-hoc assumption (ln) to explain what should be emergent from the algebra ($1/x$ fields).
4. Fail: Algebra not suitable: Since all factors in $\hat{D}\left(\prod_j \alpha_j^{\beta_j}\right)$ are octonion, the metric is Euclidean only.

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Fix (4.): Okubo algebra is a composition algebra over $\mathbb{R}^8$ that is flexible, power associative, but not alternative:

$$\exists a, b \quad : \quad a(ab) = -(aa)\, b.$$

Octonion exponentiation?

Science fiction: bridge between algebra and topology? ...

$$
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\end{aligned}
$$

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... and measurement:

$$\begin{aligned} \Phi &:= \left\{\phi : \mathbb{O} \to \mathbb{R}^4\right\}, \\ \phi_\lambda\left(\hat{D}\right) &:= \int \psi\left(\hat{D} - \lambda\right)\psi. \end{aligned}$$

Octonion exponentiation?

Science fiction: bridge between algebra and topology? ...

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Could this model an algebraic counterpart of the exotic "E_8 manifold" over $\mathbb{R}^4$?

Lattice numbers?

- Digits 1 ($\blacksquare$) and 0 ($\square$) on lattices $\rightarrow$ lattice numbers [11].
- Addition from pairwise *XOR*:

$$
\begin{array}{ccccc}
\square & \square & \square & \square & \square \\
\square & \square & \blacksquare & \square & \square \\
\square & \square & \overline{\underline{\square}} & \square & \square \\
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\end{array}
\;+\;
\begin{array}{ccccc}
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\square & \blacksquare & \square & \blacksquare & \square \\
\square & \square & \square & \square & \square
\end{array}
\;=\;
\begin{array}{ccccc}
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\square & \square & \overline{\underline{\square}} & \square & \square \\
\square & \square & \square & \blacksquare & \square \\
\square & \square & \square & \square & \square
\end{array}
$$

- Multiplication from vector addition (then *XOR*):

$$
\begin{array}{ccccc}
\square & \square & \square & \square & \square \\
\square & \square & \blacksquare & \square & \square \\
\square & \square & \overline{\underline{\square}} & \square & \square \\
\square & \blacksquare & \square & \square & \square \\
\square & \square & \square & \square & \square
\end{array}
\;+\;
\begin{array}{ccccc}
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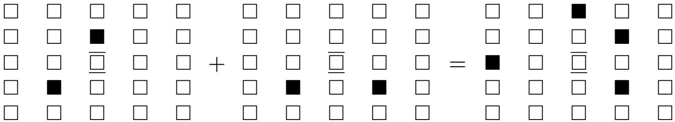
\square & \square & \square & \square & \square \\
\blacksquare & \square & \blacksquare & \square & \square
\end{array}
$$

Lattice Numbers?

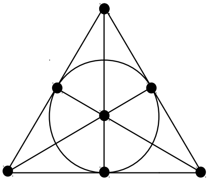
- Exponentiation from vector multiplication (then *XOR*); e.g. integral complex multiplication on the 2D square lattice:



- Integral octonion multiplication exists on the 8D square lattice and the E_8 lattice.
- Could model general 8D operation built from octonion product.
- Very much unexplored, quickly becomes unmanageable through numerical analysis.
- Very far removed from application.

Thank you!

I am grateful for all the collaboration, corrections, suggestions, patience, and help received over the years - in particular John Shuster, Geoffrey Dixon, Tevian Dray, John Huerta, and all authors and contributors credited in the references.



For Max.

Thanks for your attention!

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