Completing Some Partial Latin Squares

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Contents

Contents

Contents

Contents

Contents

Current Section

Partial latin squares

Definition 1

A partial latin square (PLS) of order n is an $n \times n$ array of n symbols in which each symbol occurs at most once in each row and column.

Partial latin squares

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Definition 2

A PLS of order n is called a latin square (LS) of order n if each cell is nonempty.

Partial latin squares

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Definition 2

A PLS of order n is called a latin square (LS) of order n if each cell is nonempty.

1		4		
2				3
	1		3	
		2		5
3				1

1	2	3	4	5
2	4	1	5	3
5	1	2	3	4
4	3	5	1	2
3	5	4	2	1

Completing PLS

Definition 3

A PLS P is called completable if there is a LS of the same order containing P.

Completing PLS

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A PLS P is called completable if there is a LS of the same order containing P.

		3		
2				
	1		3	
		5		2
3				

1	2	3	4	5
2	4	1	5	3
5	1	2	3	4
4	3	5	1	2
3	5	4	2	1

Completing PLS

When can a PLS be completed?

Completing PLS

When can a PLS be completed?

1		3		
2				3
	2	4	3	5
		5		2
3				1

Completing PLS

When can a PLS be completed?

1		3		
2				3
	2	4	3	5
		5		2
3				1

- The problem of completing PLSs is NP-complete. (Colbourn, 1984)

Completing PLS

When can a PLS be completed?

1		3		
2				3
	2	4	3	5
		5		2
3				1

- The problem of completing PLSs is NP-complete. (Colbourn, 1984)
- A good characterization of completable partial latin square is unlikely.

Equivalent Objects

A PLS P of order n is a subset of $[n] \times [n] \times [n]$ in which $(r, c, s) \in P$ if and only if symbol s occurs in cell (r, c).

Equivalent Objects

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$P =$

1		3		
2				3
	1		3	
		5		2
3				1

$(2, 1, 2), (4, 3, 5) \in P$

Equivalent Objects

A LS of order n is equivalent to a properly n-edge-colored $K_{n,n}$.

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$L =$

1	2	3
2	3	1
3	1	2

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$L =$

1	2	3
2	3	1
3	1	2

Theorem 1 (König, 1916)

Let G be a bipartite graph with $\Delta(G) = m$. Then $\chi'(G) = m$.

Isotopisms and Congujates

Let $P \in \mathrm{PLS}(n)$ and S_n be the symmetric group acting on $[n]$.

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Let $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$.

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The PLS in which the rows, columns, and symbols of P are permuted according to α, β, and γ respectively is $\theta(P) \in \mathrm{PLS}(n)$.

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The mapping θ is called an isotopism, and P and $\theta(P)$ are said to be isotopic.

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$P =$

1		3		
2				3
	1		3	
		5		2
3				1

$\theta(P) =$

3		1		
2				1
	3		1	
		5		2
1				3

Isotopisms and Congujates

Let $P \in \mathrm{PLS}(n)$ and S_n be the symmetric group acting on $[n]$.

Let $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$.

The PLS in which the rows, columns, and symbols of P are permuted according to α, β, and γ respectively is $\theta(P) \in \mathrm{PLS}(n)$.

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$P =$

1		3		
2				3
	1		3	
		5		2
3				1

$\theta(P) =$

	1	3		
	2			3
1			3	
		5		2
	3			1

Isotopisms and Congujates

The PLS in which the coordinates of each triple of P are uniformly permuted is called a conjugate of P.

Isotopisms and Congujates

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$P =$

1				
2				3
	1			
4				1

$P^{(rc)} =$

1	2			4
		1		
	3			1

Isotopisms and Congujates

The PLS in which the coordinates of each triple of P are uniformly permuted is called a conjugate of P.

$P =$

1				
2				3
	1			
4				1

$P^{(rs)} =$

1	3			5
2				
				2
5				

Isotopisms and Congujates

Theorem 2

A PLS P is completable if and only if an isotopism of P is completable.

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Theorem 3

A PLS P is completable if and only if a conjugate of P is completable.

Current Section

Hall's Theorem

Theorem 4 (Hall's Theorem, 1940)

Let $r, n \in \mathbb{Z}$ *such that* $r \leq n$. *Let* $P \in \mathrm{PLS}(n)$ *with* r *completed rows and* $n - r$ *empty rows. Then* P *can be completed to a LS of order* n.

Hall's Theorem

Theorem 4 (Hall's Theorem, 1940)

Let $r, n \in \mathbb{Z}$ such that $r \leq n$. Let $P \in \mathrm{PLS}(n)$ with r completed rows and $n - r$ empty rows. Then P can be completed to a LS of order n.

Rows can be replaced with columns or symbols.

Hall's Theorem

1	2	3	4	5	6	7
2	6	1	7	3	4	5
5	1	7	3	4	2	6

Hall's Theorem

1	2	3				
2	6	1				
3	1	7				
4	5	6				
5	7	2				
6	4	5				
7	3	4				

Hall's Theorem

1	2	3				
2		1				3
3	1	2				
			1	2	3	
	3		2	1		
			3		1	2
				3	2	1

Ryser's Theorem

Theorem 5 (Ryser's Theorem, 1950)

Let $r, s, n \in \mathbb{Z}$ such that $r, s \leq n$. Let $P \in \mathrm{PLS}(n)$ with a $r \times s$ block of symbols and empty cells elsewhere. Then P can be completed if and only if each symbol occurs $r + s - n$ times in P.

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Let $r, s, n \in \mathbb{Z}$ such that $r, s \leq n$. Let $P \in \mathrm{PLS}(n)$ with a $r \times s$ block of symbols and empty cells elsewhere. Then P can be completed if and only if each symbol occurs $r + s - n$ times in P.

1	2	3
2	4	5
5	1	2

1	2	3	7
2	4	5	6
5	1	2	4
3	5	6	1

1	2	3	5
2	4	5	6
5	1	2	4
3	5	6	1

Evans' Conjecture

Theorem 6

If $P \in \mathrm{PLS}(n)$ *with at most* $n - 1$ *non-empty cells, then* P *can be completed.*

Evans' Conjecture

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If $P \in \mathrm{PLS}(n)$ with at most $n-1$ non-empty cells, then P can be completed.

Confirmed independently by:

- Häggkvist (1979) for $n \geq 1111$

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1						
	4	5				
5						
			3			
				1		

1						
		5		4		
5						
			3			
	1					

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1					
		5		4	
5					
	1				

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7
2	7	5	1	4	6
5	4	6	2	7	1
6	5	1	7	2	4
7	6	2	4	1	5
4	1	7	6	5	2

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4	6	
5	4	6	2	7	1	
6	5	1	7	2	4	
7	6	2	4	1	5	
4	1	7	6	5	2	

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4	6	
5	4	6	2	7	1	
6	5	1	7	2	4	
7	6	2	4	1	5	
4	1	7	6	5	2	

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		2	4	7
7	6		4	1	5	2
4		7	6	5	2	1

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		2	4	7
7	6		4	1	5	2
4		7	6	5	2	1

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		7	4	2
7	6		4	1	5	2
4		7	6	5	2	1

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		7	4	2
7	6		4	2	5	1
4		7	6	5	2	1

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		7	4	2
7	6		4	2	5	1
4		7	6	1	2	5

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	3
2	7	5	1	4	3	6
5	4	6	2	3	1	7
6	5	1	3	7	4	2
7	6	3	4	2	5	1
4	3	7	6	1	2	5

There are incompletable PLSs of order n with n non-empty cells.

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1			
2			
3			
	4		

1	2	3	
			4

1			
	1		
		1	
			2

There are incompletable PLSs of order n with n non-empty cells.

1			
2			
3			
	4		

1	2	3	
			4

1			
	1		
		1	
			2

1			
2			
	3	4	

1	2		
		3	
		4	

1			
	1		
		2	
		3	

There are incompletable PLSs of order n with n non-empty cells.

1			
2			
3			
	4		

1	2	3	
			4

1			
	1		
		1	
			2

1			
2			
	3	4	

1	2		
		3	
		4	

1			
	1		
		2	
		3	

Let $B_{k,n} \in \mathrm{PLS}(n)$ with symbol 1 in the first k diagonal cells and symbols $2, 3, \ldots, n-k+1$ in the last $n-k$ cells of column $k+1$.

Theorem 7 (Andersen and Hilton, 1983)

Let $P \in \mathrm{PLS}(n)$ with exactly n non-empty cells. Then P can be completed if and only if P is not a species of $B_{k,n}$ for each $k < n$.

Current Section

One Nonempty Row, Column, and Symbol

Let $P \in \mathrm{PLS}(n)$.

One Nonempty Row, Column, and Symbol

Let $P \in \mathrm{PLS}(n)$.

If there exists r, c, and s such that for each $(x, y, z) \in P$ either $x = r$, $y = c$, or $z = s$, then P satisfies the *RCS*-property.

1	4	3	2	5	6
2	1				
3		1			
4			1		
5				1	
6					1

1	4	3	2	5	6
2	1				
3		1			
4			1		
5				1	
6					1

Casselgren and Häggkvist conjectured that if P satisfies the *RCS*-property, $(r, c, s) \in P$, and $n \notin \{3, 4, 5\}$, then P can be completed.

1	4	3	2	5	6
2	1				
3		1			
4			1		
5				1	
6					1

Casselgren and Häggkvist conjectured that if P satisfies the RCS-property, $(r, c, s) \in P$, and $n \notin \{3, 4, 5\}$, then P can be completed.

They confirmed (2013) $n \in \{6, 7\}$ and $n = 4k$ for all $k \geq 2$.

1	2	3
2	1	
3		1

1	3	4	2
2	1		
3		1	
4			1

1	3	2	4	5
2	1			
3		1		
4			1	
5				1

2	3		4	5
	1			
3		1		
4			1	
5				1

Theorem 8 (Kuhl and Schroeder, 2016)

Let $P \in \mathrm{PLS}(n)$ satisfy the RCS-property. If $n \notin \{3, 4, 5\}$ and P does not contain a species of $B_{k,n}$ for each $k \in [n-1]$, then a completion of P exists.

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	7	2	6	3	4
2	1					
5		1				
3			1			
4				1		
6					1	
7						1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	2	7	6	3	4
2	1					
5			1			
3		1				
4				1		
6					1	
7						1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	2	7	6	3	4
2	1					
5		1				
3			1			
4				1		
6					1	
7						1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	2	7	6	3	4
2	1					
5		1	4			
3		4	1			
4				1		
6					1	
7						1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	2	7	6	3	4
2	1					
5		1	4			
3		4	1			7
4				1	6	
6				4	1	
7			3			1

1	5	2
2	1	
5		1

7	6	3	4
4			
3	4	6	7

3		4	7
4			6
6			4
7			3

1			7
	1	6	
	4	1	
3			1

1	5	2
2	1	5
5	2	1

7	6	3	4
6	7	4	3
4	3	7	6
3	4	6	7

3	6	4	7
4	7	3	6
6	3	7	4
7	4	6	3

1	2	5	7
2	1	6	5
5	4	1	2
3	5	2	1

1	5	2	7	6	3	4
2	1	5	6	7	4	3
5	2	1	4	3	7	6
3	6	4	1	2	5	7
4	7	3	2	1	6	5
6	3	7	5	4	1	2
7	4	6	3	5	2	1

1	5	2	7	6	3	4
2	1	5	6	7	4	3
5	2	4	1	3	7	6
3	6	1	4	2	5	7
4	7	3	2	1	6	5
6	3	7	5	4	1	2
7	4	6	3	5	2	1

One Nonempty Row, Column, and Symbol

4	5	2	6	7	3	1
2					1	
3				1		
7			1			
5		1				
6	1					
1						

Completed Rows and Columns

When can a PLS with exactly *a* rows and *b* columns be completed?

Completed Rows and Columns

When can a PLS with exactly *a* rows and *b* columns be completed?

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

Completed Rows and Columns

When can a PLS with exactly *a* rows and *b* columns be completed?

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

- Buchanan solved problem for $a = b = 2$ in dissertation (2007)

Completed Rows and Columns

When can a PLS with exactly *a* rows and *b* columns be completed?

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

- Buchanan solved problem for $a = b = 2$ in dissertation (2007)
- Adam, Bryant, and Buchanan shortened dissertation (2008)

Completed Rows and Columns

When can a PLS with exactly *a* rows and *b* columns be completed?

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

- Buchanan solved problem for $a = b = 2$ in dissertation (2007)
- Adam, Bryant, and Buchanan shortened dissertation (2008)
- Kuhl and McGinn proved same result and more (2017)

Completed Rows and Columns

$Y =$

1	2	3	4
3	4	2	1
2	3		
4	1		

$Z =$

1	2	3	4	5
3	1	2	5	4
2	3			
4	5			
5	4			

Completed Rows and Columns

$Y =$

1	2	3	4
3	4	2	1
2	3		
4	1		

$Z =$

1	2	3	4	5
3	1	2	5	4
2	3			
4	5			
5	4			

Let Γ denote the set of all isotopisms of Y and Z.

Theorem 9

Let $n \geq 2$ and $A \in \mathrm{PLS}(2, 2; n)$. The partial latin square A can be completed if and only if $A \notin \Gamma$.

Completed Rows and Columns

There is a symbol not in an intercalate.

Completed Rows and Columns

There is a symbol not in an intercalate.

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3
2	7	1	3	6	5
7	3				
6	5				
3	6				
5	1				

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3
2	7	1	3	6	5
7	3	2	1	5	6
6	5	3	7	2	1
3	6	7	5	1	2
5	1	6	2	3	7

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	
2	7	1	3	6	5	
7	3	2	1	5	6	
6	5	3	7	2	1	
3	6	7	5	1	2	
5	1	6	2	3	7	

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	
2	7	1	3	6		5
7	3	2	1		6	5
6	5	3		2	1	7
3	6		5	1	2	7
5		6	2	3	7	1

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	
2	7	1	3	6		5
7	3	2	1		5	6
6	5	3		2	1	7
3	6		7	5	2	1
5		6	2	1	7	3

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3	2	1	4	5	6
6	5	3	4	2	1	7
3	6	4	7	5	2	1
5	4	6	2	1	7	3

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3	2	1	4	5	6
6	5	3	4	2	1	7
3	6	4	7	5	2	1
5	4	6	2	1	7	3
4	1	7	5	3	6	2

Completed Rows and Columns

Each symbol is in an intercalate.

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

1	2	3	4	6	5
2	3	1	5	4	6
3	1				
6	4				
5	6				
4	5				

Completed Rows and Columns

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

1	2	3	4	6	5
2	3	1	5	4	6
3	1	4	6	5	2
6	4	5	1	2	3
5	6	2	3	1	4
4	5	6	2	3	1

Completed Rows and Columns

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
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4	3	1	6	5	2	
3	1	2	4	6	5	
1	2	3	5	4	6	
5	6	4	1	2	3	
2	5	6	3	1	4	
6	4	5	2	3	1	

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1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

4	3	1	6	5	2	
3	1	2	4	6		5
1	2	3	5		6	4
5	6	4		2	3	1
2	5		3	1	4	6
6		5	2	3	1	4

Completed Rows and Columns

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

4	3	1	6	5	2	
3	1	2	4	6		5
1	2	3	5		6	4
5	6	4		2	3	1
2	5		3	1	4	6
6		5	2	4	1	3

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1	2	3	4	5	6	7
2	3	1	5	4	7	6
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3	1	2	4	6	7	5
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5	6	4	7	2	3	1
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6	7	5	2	4	1	3
7	4	6	1	3	5	2

Theorem 10 (Kuhl and McGinn, 2017)

Let $A \in \mathrm{PLS}(2, b; n)$ and cells $[2] \times [b]$ consist only of symbols from $[b]$. If $n \geq 2b^2 - 2b + 5$ and $\sigma_A([n] \backslash [b])$ contains a cycle of length at least $\frac{n+3}{2}$, then A can be completed.

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Conjecture 1

Let $A \in \mathrm{PLS}(2, b; n)$*. If* $n \geq 2b + 2$*, then* A *can be completed.*

Current Section

Häggkvist Conjecture

Conjecture 2 (Häggkvist, 1979)

If $P \in \mathrm{PLS}(nr)$ with all non-empty cells in at most $n-1$ pairwise disjoint $r \times r$ blocks, then P can be completed.

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- $n = 3$ was solved by Denly and Häggkvist (2003)
- Kuhl and Denley confirmed Conjecture 1 for latin $r \times r$ blocks (2008)

Block Diagonal

Theorem 11 (Kuhl and Schroeder, 2015)

Let n and r be positive integers.

- *If* $n \geq r+1$, *then for every* $A \in \mathrm{LS}(r;[nr])$, *nA is completable.*
- *If* $n \leq r-1$, *then there exists* $A \in \mathrm{LS}(r;[nr])$ *for which nA is not completable.*

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1	2				
2	3				
		1	2		
		2	3		
				1	2
				2	3

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Conjecture 3

Let n and r be positive integers. If $n \geq r$, *then for every* $A \in \mathrm{LS}(r; [nr])$, *nA is completable.*

Disjoint Subsquares

$\mathrm{PLS}(a^s, b^t)$: PLSs with $s + t$ pairwise disjoint subsquares, where s subsquares have order a and t subsquares have order b.

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1	2							
2	1							
		3	4					
		4	3					
				5	6			
				6	5			
						7	8	9
						8	9	7
						9	7	8

Disjoint Subsquares

Theorem 13 (Heinrich, 1982)

- *Each element of* $\mathrm{PLS}(a, b, c)$ *is completable if and only if* $a = b = c$*.*
- *Each element of* $\mathrm{PLS}(a, b, c, d)$ *is completable if and only if* $a = b = c$ *and* $d \leq 2a$*.*

Disjoint Subsquares

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- *Each element of* $\mathrm{PLS}(a, b, c, d)$ *is completable if and only if* $a = b = c$ *and* $d \leq 2a$.

Theorem 14 (Heinrich, 1982)

Suppose that $a < b$.

- *If* $s \geq 3$ *and* $t \geq 3$*, then each element of* $\mathrm{PLS}(a^s, b^t)$ *is completable.*
- *Each element of* $\mathrm{PLS}(a, b^t)$ *is completable if and only if* $t \geq 3$.
- *Each element of* $\mathrm{PLS}(a^s, b)$ *is completable if and only if* $(s-1)a \geq b$.

Disjoint Subsquares

Theorem 15 (Kuhl and Schroeder, 2017)

Suppose that $a < b$.

- *Each element of* $\mathrm{PLS}(a^2, b^t)$ *is completable if and only if* $t \geq 3$.
- *Each element of* $\mathrm{PLS}(a^s, b^2)$ *is completable if and only if* $as \geq b$.

Disjoint Subsquares

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Suppose that $a < b$.

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Problems:

- Find conditions on s, t, and u that guarantee completions of the elements of $\mathrm{PLS}(a^s, b^t, c^u)$.

Disjoint Subsquares

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Problems:

- Find conditions on s, t, and u that guarantee completions of the elements of $\mathrm{PLS}(a^s, b^t, c^u)$.
- Classify the completable elements of $\mathrm{PLS}(a, b, c, d, e)$.

Diagonally Cyclic Latin Squares

Definition 4

A LS L is diagonally cyclic if for each $(i, j, k) \in L$, $(i+1, j+1, k+1) \in L$.

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0	2	4	1	3
4	1	3	0	2
3	0	2	4	1
2	4	1	3	0
1	3	0	2	4

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- A diagonally cyclic LS is determined by its first row.

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0	2	4	1	3
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- Suppose that $(0, i, s_i) \in L$. If $s_i - i \not\equiv s_j - j$ for each i, j, then L is diagonally cyclic.

Diagonally Cyclic Latin Squares

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0	2	4	1	3
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- A diagonally cyclic LS is determined by its first row.
- Suppose that $(0, i, s_i) \in L$. If $s_i - i \neq s_j - j$ for each i, j, then L is diagonally cyclic.
- There are no diagonally cyclic LSs of even order.

Diagonally Cyclic Latin Squares

Let $P \in \mathrm{PLS}(n)$ with k diagonals completed cyclically. Can P be completed to a diagonally cyclic LS?

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0	2							1

Diagonally Cyclic Latin Squares

0	2							1

0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0
2	3	4	5	6	7	8	0	1
3	4	5	6	7	8	0	1	2
4	5	6	7	8	0	1	2	3
5	6	7	8	0	1	2	3	4
6	7	8	0	1	2	3	4	5
7	8	0	1	2	3	4	5	6
8	0	1	2	3	4	5	6	7

Diagonally Cyclic Latin Squares

0	2	7	6	8	4	3	5	1

0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0
2	3	4	5	6	7	8	0	1
3	4	5	6	7	8	0	1	2
4	5	6	7	8	0	1	2	3
5	6	7	8	0	1	2	3	4
6	7	8	0	1	2	3	4	5
7	8	0	1	2	3	4	5	6
8	0	1	2	3	4	5	6	7

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Let $N(k)$ be the smallest integer in which all PLSs of odd order $n \geq N(k)$ with k cyclic diagonals can be completed cyclically.

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Conjecture 4

$N(3) = 9$.

ϵ-dense PLSs

Let $P \in \mathrm{PLS}(n)$. We say that P is ϵ-dense if each row, column, and symbol is used at most ϵn times.

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Theorem 16 (Daykin and Häggkvist, 1984)

If $n = 16k$, then all $\frac{1}{2^9\sqrt{n}}$-dense PLSs of order n are completable.

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If $n = 16k$, then all $\frac{1}{2^9\sqrt{n}}$-dense PLSs of order n are completable.

Theorem 17 (Bartlett, 2013)

All 10^{-4}-dense PLSs of order n are completable for $n > 1.2 \times 10^5$.

Current Section

Other Completion Problems

Conjecture 6

If P is a partial latin cube of order n with at most $n - 1$ non-empty cells, then P can be completed to a latin cube of order n.

Theorem 18 (Kuhl and Denley, 2011)

If P is a partial latin cube of order n with at most $n - 1$ non-empty cells, no two of which lie in the same row, then P can be completed to a latin cube of order n.

Other Completion Problems

Conjecture 7

Let $P \in \mathrm{PLS}(n)$ with at most $n-1$ non-empty cells. Let $\mathcal{Q} \subseteq \mathrm{PLS}(n)$ be the PLSs that avoid P. For any $Q \in \mathcal{Q}$, P can be completed to a LS that avoids Q.

Theorem 19 (Kuhl and Denley, 2012)

Let $P \in \mathrm{PLS}(4k)$ with at most $k-1$ non-empty cells. Let $\mathcal{Q} \subseteq \mathrm{PLS}(n)$ be the PLSs that avoid P. For any $Q \in \mathcal{Q}$, P can be completed to a LS that avoids Q.

Other Completion Problems

Conjecture 8

Any two PLSs of order $n > 5$ can be avoided simultaneously.

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All PLSs of order $n \geq 4$ are avoidable.

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Any two PLSs of order $n > 5$ can be avoided simultaneously.

Theorem 20 (Chetwynd and Rhodes; Cavenagh; Kuhl and Denley)

All PLSs of order $n \geq 4$ are avoidable.

Theorem 21 (Kuhl and Hinojosa, 2012)

- *Any two PLSs of order $4k$ with $k > 56$ can be avoided simultaneously.*

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- *Any two PLSs of order mk with $k \geq \frac{m^5}{2}$ can be avoided simultaneously.*

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Conjecture 6

Let $P_1, \ldots, P_t \in \mathrm{PLS}(n)$. If $t < n/3$, then $P_1, \ldots, P_t$ can be avoided simultaneously.

Thank You!