Modules over semisymmetric quasigroups

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Motivation: Group Modules

- A right $G$-module $M$ is an abelian group equipped with homomorphism
  \[ \rho : G \rightarrow \text{Aut}(M). \]

- Split extension $G \rtimes^\rho M$ built up on $G \times M$ by
  \[ (g, a)(h, b) = (gh, a \cdot h + b). \]

- The projection $\pi : G \rtimes^\rho M \rightarrow G$ is an abelian group object in $\text{Gp}/G$. 
1. Semisymmetry

2. Multiplication Groups: Combinatorial and Universal

3. Modules over General Quasigroups

4. Modules over Semisymmetric Quasigroups
Semisymmetry
A semisymmetric quasigroup \((Q, \cdot, /, \backslash)\) satisfies the following equivalent identities

\[
(yx)y = x; \\
y(xy) = x; \\
y \backslash x = xy; \\
y / x = xy.
\]

- The divisions / and \(\backslash\) coincide and are the opposite of multiplication.
- The class of semisymmetric quasigroups forms a variety \(\mathbf{P}\).
Mendelsohn (cyclic) triple systems $\leftrightarrow$ semisymmetric, idempotent quasigroups

**Theorem: Mendelsohn (1971)**
An MTS exists for all $n \neq 2 \mod 3$, except for $n = 1$ and $n = 6$.

**Corollary**
Let $n \neq 2 \mod 3$ be a positive integer. If $n \neq 6$, then there is a semisymmetric, idempotent quasigroup of order $n$. 
• Given a quasigroup \((Q, \cdot, /, \backslash)\), we can define a semisymmetric quasigroup on \(Q^3\) by

\[
(x_1, x_2, x_3)(y_1, y_2, y_3) = (y_3/x_2, y_1\backslash x_3, x_1 \cdot y_2).
\]

• This yields an adjunction \(P \leftrightarrow Qtp\) (Smith, 1997).
Multiplication Groups: Combinatorial and Universal
Relative Multiplication Groups

- Let $\text{Mlt}(Q) \leq Q!$ denote the combinatorial multiplication group, or the group of permutations of $Q$ generated by the set of left and right multiplications $\{L(q), R(q) \mid q \in Q\}$.

**Definition: Relative Multiplication Group**

Given a subquasigroup $P$ of $Q$, the group of permutations of $Q$ generated by

$$\{L(p), R(p) \mid p \in P\},$$

denoted $\text{Mlt}_Q(P)$, is the relative multiplication group of $P$ in $Q$. 
Universal Multiplication Groups

- Let $V$ be a variety of quasigroups.
- Let $Q[X]$ be the coproduct in $V$ of $Q$ with the free $V$-quasigroup on the singleton $\{X\}$.
- As a quasigroup, $Q[X]$ is isomorphic to the free $V$-extension of the partial Latin square furnished by the multiplication table of $Q$.

**Definition: Universal Multiplication Group**

The universal multiplication group $U(Q; V) := \text{Mlt}_{Q[X]}(Q)$, also denoted $\tilde{G}$, of $Q$ in $V$ is the relative multiplication group of $Q$ in $Q[X]$.

- This establishes a functor $U(\quad; V) : V \rightarrow \text{Gp}$.
- In $Q$, $\tilde{G}$ is free on the disjoint union $\{\tilde{L}(q) \mid q \in Q\} + \{\tilde{R}(q) \mid q \in Q\}$ of two copies of $Q$ (Smith 1986).
The Semisymmetric Case

Proposition

Let \( Q \) be a semisymmetric quasigroup. Then \( U(Q; P) \) is the free group on the set \( \{ \tilde{R}(q) \mid q \in Q \} \).

Proof Sketch:

- Map \( R : QG \to \tilde{G} ; q_1^{\varepsilon_1} \cdots q_n^{\varepsilon_n} \mapsto \tilde{R}(q_1)^{\varepsilon_1} \cdots \tilde{R}(q_n)^{\varepsilon_n} \).
- Showing \( R \) is injective comes down to showing that \( X \tilde{R}(q_1)^{\varepsilon_1} \cdots \tilde{R}(q_n)^{\varepsilon_n} \) is an irreducible word in \( Q[X] \).
• Notice that $\tilde{G}$ acts transitively on the set $Q$; for example,

$$x\tilde{R}(q) = xR(q) = x \cdot q.$$ 

**Definition: Universal Stabilizer**

Let $\tilde{G} = U(Q; V)$ be the universal multiplication group of a quasigroup containing the element $e \in Q$. Then the stabilizer of $e$ under the action of $\tilde{G}$ on $Q$, denoted $\tilde{G}_e$, is called the universal stabilizer of $Q$ in $V$.

• In $Q$, $\tilde{G}_e$ is free on

$$\{\tilde{R}(e \backslash q)\tilde{L}(q \slash e)^{-1}, \tilde{R}(e \backslash q)\tilde{R}(r)(e \backslash qr)^{-1}, \tilde{L}(q \slash e)\tilde{L}(r)\tilde{L}(rq \slash e)^{-1} \mid q, r \in Q\}$$

(Smith 1986).
Let's take a graphical/topological approach to $\tilde{G}_e$ in $P$:

\begin{itemize}
  \item **Theorem:** In $P$, $\tilde{G}_e$ is free on
  \[
  \{ R(e^2), R(xe)R(ex), R(xe)R(y)R(xy\cdot e)^{-1} \mid (x, y) \in Q - \{e\} \times Q, y \neq xe \}.
  \]
\end{itemize}
Modules over General Quasigroups
Definition: Quasigroup Module (Smith, 1986)

Let $Q$ be a nonempty quasigroup (containing some point $e$) in the variety $Q$ with universal multiplication group $\tilde{G} = U(Q; Q)$. We define $Q$-modules to be modules, in the group theoretic sense, over the universal stabilizer $\tilde{G}_e$.

- $Q$-modules are just $\mathbb{Z}\tilde{G}_e$-modules.
Quasigroup Structure on $\tilde{G}_e$-modules

- Let $Q$ be a quasigroup in $Q$, and let $M$ be a $\tilde{G}_e$-module.
- In the spirit of split extensions, let $E = M \times Q$, and $\pi : E \to Q$ be the projection onto $Q$.
  - Now $\tilde{G}_e$ acts on $\pi^{-1}\{e\}$, a local copy of $M$ in $E$ over $e \in Q$.
- Moreover,
  \[
  \begin{align*}
    a \cdot b &= a\tilde{R}(b^\pi) + b\tilde{L}(a^\pi), \\
    a/b &= (a - b\tilde{L}(a^\pi/b^\pi))\tilde{R}(b^\pi)^{-1}, \\
    a \backslash b &= (b - a\tilde{R}(a^\pi/b^\pi))\tilde{L}(a^\pi)^{-1},
  \end{align*}
  \]
  furnishes $E$ with a quasigroup structure.
Modules over Semisymmetric Quasigroups
• How can we describe modules over quasigroups in $\mathbf{P}$?
• We want to take $\mathbb{Z}\tilde{G}_e$, and modulo out by an ideal which accounts for the identity $(yx)y = x$. 
Example: Semisymmetric Modules over \( \{e\} \)

- **Note** \( \mathbb{Z}\tilde{G}_e \cong \mathbb{Z}[X, X^{-1}] \).
- Define \( x \cdot y = x\tilde{R}(y^\pi) + y\tilde{R}(x^\pi)^{-1} = x\tilde{R}(e) + y\tilde{R}(e)^{-1} \) for all \( x, y \in \mathbb{Z}\tilde{G}_e \).
- Then \( x = (yx)y \implies x = y\tilde{R}(e)\tilde{R}(e) + x\tilde{R}(e)^{-1}\tilde{R}(e) + y\tilde{R}(e)^{-1} \).
- Set \( x = 0 \), and we find \( 0 = y\tilde{R}(e)^2 + y\tilde{R}(e)^{-1} \iff 0 = y\tilde{R}(e)^3 + y \) for all \( y \in \mathbb{Z}\tilde{G}_e \).
- **Semisymmetric quasigroup modules over \( \{e\} \)** are modules over \( \mathbb{Z}\tilde{G}_e/(\tilde{R}(e)^3 + 1) \cong \mathbb{Z}[X, X^{-1}]/(X^3 + 1) \).
### Theorem

Let $Q$ be a quasigroup in the variety $\mathbf{P}$ of semisymmetric quasigroups. The category of $Q$-modules in $\mathbf{P}/Q$ is equivalent to the category of modules over $\mathbb{Z}\tilde{G}_e/J$, where $J$ is the two-sided ideal generated by

$$\{\tilde{R}(ye)(\tilde{R}(x)\tilde{R}(y) + \tilde{R}(yx)^{-1})\tilde{R}(xe)^{-1} \mid x, y \in Q\}.$$
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