DISTRIBUTIVE ALGEBRAS AND THE AXIOMATIZATION OF CONVEXITY

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OUTLINE

- Convex sets and barycentric algebras:
basic definitions, examples and properties

- Threshold convexity
and the problem of axiomatization of convexity

- Threshold barycentric algebras
and their varieties

- Beyond threshold barycentric algebras

BARYCENTRIC ALGEBRAS

$\mathbb{R}$ - the field of reals;
$I^\circ :=]0,1[= (0,1) \subset \mathbb{R}$.

Barycentric algebra - an algebra $(A, \underline{I}^\circ)$, with a binary operation $\underline{p}$ for each operator $p \in I^\circ$, axiomatized by the following:

idempotence (I): $xx\underline{p} = x$,

skew-commutativity (SC):
$xy\underline{p} = xy\underline{1-p} =: xy\underline{p}'$,

skew-associativity (SA):
$[xy\underline{p}]\, z\, \underline{q} = x\,[yz\underline{q/(p\circ q)}]\,\underline{p\circ q}$

for all $p, q \in I^\circ$, where
$p \circ q = (p'q')' = p + q - pq$.

Skew associativity is also written as:

$$[xy\underline{p}]z\underline{q} = x[\,yz\underline{(p \circ q \to q)}\,]\,\underline{p \circ q}$$

where

$$p \to q = \begin{cases} 1 & \text{if } p = 0; \\ q/p & \text{otherwise} \end{cases}$$

Proposition The class $\mathcal{B}$ of barycentric algebras is a variety.

$\mathcal{B}$ also satisfies:

entropicity (E):
$[xy\underline{p}]\,[zt\underline{p}]\,\underline{q} = [xz\underline{q}]\,[yt\underline{q}]\,\underline{p}$

and

distributivity (D):

$[xy\underline{p}]\,z\,\underline{q} = [xz\underline{q}]\,[yz\underline{q}]\,\underline{p}$,
$x\,[yz\underline{p}]\,\underline{q} = [xy\underline{q}]\,[xz\underline{q}]\,\underline{p}$,

for all $p, q \in I^\circ$.

EXAMPLES

• **Convex subsets** of real spaces under the operations

$$xy\underline{p} = xp' + yp = x(1-p) + yp$$

for each $p \in I^\circ$.

Convex sets form the subquasivariety $\mathcal{C}$ of the variety $\mathcal{B}$ defined by the cancellation laws

$$(xy\underline{p} = xz\underline{p}) \to (y = z)$$

for all operations $\underline{p}$ of $\underline{I}^\circ$.

• **"Stammered" semilattices** $(S, \cdot)$ with the operation $x \cdot y = xy\underline{p}$ for all $p \in I^\circ$.
They form the subvariety $\mathcal{SL}$ of $\mathcal{B}$ defined by $xy\underline{p} = xy\underline{r}$ for all $p, r \in I^\circ$.

• Certain **sums of convex sets** over semilattices.

THEOREM Each barycentric algebra is a subalgebra of a Płonka sum of convex sets over its semilattice replica.

EXTENDED BARYCENTRIC ALGEBRAS

Barycentric algebras may be considered as **extended barycentric algebras** $(A, \underline{I})$, where $I = [0,1] \subset \mathbb{R}$, and with the operations $\underline{0}$ and $\underline{1}$ defined by

$$xy\underline{0} = x \text{ and } xy\underline{1} = y.$$

Proposition The class $\overline{\mathcal{B}}$ of extended barycentric algebras is a variety, specified by the identities (I), (SC), (SA) and the two above.

Examples: convex sets and semilattices considered as usual barycentric algebras, with two additional operations $\underline{0}$ and $\underline{1}$.

THRESHOLD CONVEXITY

Given a real number $0 \leq t \leq 1/2$, known as the **threshold**.

For elements x, y of a convex set C, define

$$xy\,\underline{\underline{r}} = \begin{cases} x & \text{if } r < t; \\ xy\,\underline{r} = x(1-r) + yr & \text{if } t \leq r \leq t'; \\ y & \text{if } r > t' \end{cases}$$

for $0 < r < 1$. Then the binary operations $\underline{\underline{r}}$ are described as **threshold-convex combinations** (**small, moderate** and **large**).

Let $\underline{\underline{I}}^{\circ} = \{\underline{\underline{r}} \mid r \in I^{\circ}\}$.

Proposition Convex sets $(C, \underline{\underline{I}}^{\circ})$ are idempotent, skew-commutative and entropic algebras.

KEIMEL'S QUESTION

Keimel's question: Can the skew associativity be replaced by the entropic law?

THEOREM In the specification of barycentric algebras, the axiom of skew-associativity cannot be replaced by the axiom of entropicity.

Counterexample is given for $t = 1/2$ by the algebra $(I, \underline{\underline{I}}^{\circ})$, which is not skew-associative.

For $p = q = 1/2$,
$p \circ q = 3/4$ and $p \circ q \to q = 2/3$.
Then for $x = y = 0$,
$(xy\underline{1/2})z\,\underline{1/2} = 1/2$ and
$x[yz(\underline{\underline{(p \circ q) \to q}})]\,\underline{\underline{p \circ q}} = 1$.

THRESHOLD BARYCENTRIC ALGEBRAS

Given a threshold $0 \leq t \leq 1/2$.
$\mathcal{B}^t$ - the variety of **threshold-t barycentric algebras**, generated by the convex sets $(C, \underline{\underline{I}}^\circ)$.

Examples:

- $\mathcal{B}^0 = \mathcal{B}$
- $\mathcal{B}^{1/2}$

with $\underline{\underline{r}} = \underline{0}$ for $r < 1/2$ and $\underline{\underline{r}} = \underline{1}$ for $r > 1/2$

Proposition $\mathcal{B}^{1/2}$ is equivalent to the variety $\overline{\mathcal{CBM}}$ of commutative binary modes (commutative idempotent entropic groupoids) extended by the trivial operations $\underline{0}$ and $\underline{1}$.

- $\mathcal{B}^t$ for $t \neq 0, 1/2$

with $\underline{\underline{r}} = \underline{r}$ for $t \leq r \leq t'$,
$\underline{\underline{r}} = \underline{0}$ for $r < t$ and
$\underline{\underline{r}} = \underline{1}$ for $r > t'$.

THEOREM The variety $\mathcal{B}^t$ is skew-associative if and only if $t = 0$.

SIMPLICES

Δ_k - the real k-dimensional simplex
- the free barycentric algebra in $\mathcal{B}$
over the set $X = \{x_0, \ldots, x_k\}$ of its vertices.

Proposition Let $0 < t < 1/2$ be a threshold. Then the closed unit interval $(I, \underline{\underline{I}}^\circ)$ is generated by $\{0, 1\}$.

THEOREM Let $0 < t < 1/2$ be a threshold. Then each simplex $(\Delta_k, \underline{\underline{I}}^\circ)$ is generated by its vertices.

Corollary Each simplex Δ_k is generated by its vertices under the moderate threshold-convex combinations.

VARIETIES

THEOREM Set a threshold $0 < t < 1/2$. Then each variety $\mathcal{B}^t$ is equivalent to the variety $\overline{\mathcal{B}}$ of extended barycentric algebras.

THEOREM For a threshold $0 < t < 1/2$, the variety $\mathcal{B}^t$ is defined by the following identities:

(a) Idempotence, skew-commutativity and entropicity for all operations of $\underline{\underline{I}}^{\circ}$;
(b) The identity $xy\,\underline{\underline{r}} = x$ for all $r < t$;
(c) The identity $xy\,\underline{\underline{r}} = y$ for all $r > t'$;
(d) Skew-associativity for (derived) binary operations generated by the moderate operations $\underline{\underline{p}}$ for $t \le p \le t'$.

THRESHOLD SEMILATTICES

Proposition The variety $\mathcal{B}$ of barycentric algebras contains only one proper non-trivial subvariety, namely the variety $\mathcal{SL}$ of (stammered) semilattices.

Proposition Let $0 \leq t \leq 1/2$ be a threshold. Then the variety $\mathcal{B}^t$ of threshold-t barycentric algebras contains only one proper non-trivial subvariety, namely the variety $\mathcal{SL}^t$ of threshold-t semilattices.

For each $t > 0$, the variety $\mathcal{SL}^t$ is defined by:
- equality between the respective moderate operations $\underline{\underline{p}}$, for $t \leq p \leq t'$;
- associativity for each moderate operation $\underline{\underline{p}}$ for $t \leq p \leq t'$.

$\mathcal{SL}^t$ is equivalent to the variety $\overline{\mathcal{SL}}$ of extended semilattices.

BEYOND THRESHOLD BARYCENTRIC ALGEBRAS

Set thresholds $0 \leq s < t \leq 1/2$.

$\mathcal{B}^{s,t}$ - the variety of idempotent, entropic, skew-commutative $\underline{\underline{I}}^{\circ}$-algebras defined by the following identities:

- $xy\,\underline{\underline{p}} = x$ for all $p < s$;
- $xy\,\underline{\underline{p}} = y$ for all $p > s'$;
- all identities true in the reducts $(C, \underline{\underline{[t, t']}})$ of threshold-t convex sets $(C, \underline{\underline{I}}^{\circ})$ with respect to moderate threshold combinations.

THEOREM For $t \in [0, 1/2]$, the varieties $\mathcal{B}^t$ of threshold-t barycentric algebras form an antichain.

- The meet of any two of them is the trivial variety $\mathcal{T}$.
- For thresholds $0 \leq s < t \leq 1/2$, the join $\mathcal{B}^s \vee \mathcal{B}^t$ of the varieties $\mathcal{B}^s$ and $\mathcal{B}^t$ is equal to the variety $\mathcal{B}^{s,t}$.

$\mathbf{L}(\mathcal{V})$ -
the lattice of subvarieties of a variety $\mathcal{V}$.

Then $\mathbf{L}(\mathcal{B}^{1/2}) \cong \mathbf{L}(\mathcal{CBM})$;
$\mathbf{L}(\mathcal{CBM}) \setminus \mathcal{CBM} \cong \mathbf{2} \times (\mathbb{N}, |)$.
Each proper subvariety of $\mathcal{CBM}$ is defined by one binary identity.

Corollary Let $0 \leq s, t, u, w \leq 1/2$ be thresholds, and let $s < t$ and $u < w$. Then the variety $\mathcal{B}^{s,t}$ is a subvariety of $\mathcal{B}^{u,w}$ if and only if $u \leq s < t \leq w$.

Corollary The join of the varieties $\mathcal{B}^{s,t}$ is the variety $\mathcal{B}^{0,1/2}$, equivalent to the variety $\mathcal{CBM}$ of commutative binary modes.

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