Generalized Dihedral Automorphic Loop and its Half-isomorphisms

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Inner mapping group

Let $L$ be a loop and $x \in L$. We define the left and right translations of $x$ in $L$, respectively by:

$$(y)\mathcal{L}_x = xy \quad (y)\mathcal{R}_x = yx \quad (y \in L)$$
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$$(y)L_x = xy \quad \quad (y)R_x = yx \quad \quad (y \in L)$$

The *multiplication group* of $L$ is the set:

$$\text{Mlt}(L) = \langle L_x, R_x \mid x \in L \rangle$$
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The multiplication group of $L$ is the set:

$$Mlt(L) = \langle L_x, R_x \mid x \in L \rangle$$

The inner mapping group of $L$ is defined by:

$$Inn(L) = \{ \varphi \in Mlt(L) \mid (1)\varphi = 1 \}$$
A loop $L$ is called *automorphic loop*, or *A-loop*, if all elements of $\text{Inn}(L)$ are automorphisms of $L$. 

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1 Kinyon, Kunen, Phillips, Vojtechovsky; The Structure of Automorphic Loops
Automorphic Loop

A loop $L$ is called automorphic loop, or $A$-loop, if all elements of $\text{Inn}(L)$ are automorphisms of $L$.

In the paper The Structure of Automorphic Loops$^1$, the authors constructed a type of automorphic loop from a group. They called it generalized dihedral automorphic loop.

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Let $G$ be a finite abelian group and $\alpha \in Aut(G)$. Define $Dih(\alpha, G) = \mathbb{Z}_2 \times G$ with the following operation:

$$(i, u) \ast (j, v) := (i + j, \alpha^{ij}(u^{-1})^j v)) \quad i, j \in \mathbb{Z}_2, u, v \in G$$
Let $G$ be a finite abelian group and $\alpha \in Aut(G)$. Define $Dih(\alpha, G) = \mathbb{Z}_2 \times G$ with the following operation:

$$(i, u) \ast (j, v) := (i + j, \alpha^{ij}(u^{(-1)^j}v)) \quad i, j \in \mathbb{Z}_2, u, v \in G$$

or, for $u, v \in G$:

$$(0, u) \ast (0, v) = (0, uv)$$
$$(0, u) \ast (1, v) = (1, u^{-1}v)$$
$$(1, u) \ast (0, v) = (1, uv)$$
$$(1, u) \ast (1, v) = (0, \alpha(u^{-1}v))$$
In the same paper \(^2\), the authors proved that \( \text{Dih}(\alpha, G) \) is an automorphic loop.

\(^2\) The Structure of Automorphic Loops
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In the same paper \(^2\), the authors proved that \(Dih(\alpha, G)\) is an automorphic loop.

It is easy to see that if \(\alpha = I_d\), then \(Dih(\alpha, G)\) is a group, and vice versa.

Also, \(Dih(\alpha, G)\) is commutative if, and only if, \(G\) has period 2.

\(^2\) The Structure of Automorphic Loops
Let $(L, \ast)$ and $(L', \cdot)$ be loops.
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A **half-isomorphism** of \((L, \ast)\) into \((L', \cdot)\) is a bijection \(f : L \rightarrow L'\)

\[
f(x \ast y) = f(x) \cdot f(y)
\]

or

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for all \(x, y \in L\).
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A half-isomorphism is *trivial* if it is either an isomorphism or an anti-isomorphism.
Question 1. Are there non trivial half-isomorphisms between generalized dihedral automorphic loops?
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Yes. Using the software GAP, we found many examples of non trivial half-isomorphisms between generalized dihedral automorphic loops of order 6, 8, 10, etc.
Question 2. What are the conditions for the existence of a non trivial half-isomorphism of $Dih(\alpha, G)$ into $Dih(\beta, G)$?
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**Question 3.** How many non trivial half-isomorphisms does exist of $Dih(\alpha, G)$ into $Dih(\beta, G)$?
Questions

Question 2. What are the conditions for the existence of a non trivial half-isomorphism of $Dih(\alpha, G)$ into $Dih(\beta, G)$?

Question 3. How many non trivial half-isomorphisms does exist of $Dih(\alpha, G)$ into $Dih(\beta, G)$?

Question 4. What form do these non trivial half-isomorphisms take?
Lemma 1. Let \((L, \ast)\), \((L', \cdot)\) be loops and \(f : L \to L'\) a half-isomorphism. If \(L'\) is commutative, then \(L\) is commutative and \(f\) is an isomorphism.

Proposition 1. Let \(f : \text{Dih}(\alpha, G) \to \text{Dih}(\beta, G)\) be a half-isomorphism. If \(\text{Dih}(\alpha, G)\) or \(\text{Dih}(\beta, G)\) is commutative, then \(f\) is an isomorphism.

From now on we assume that \(G\) does not have period 2.

Lemma 2. Let \((L, \ast)\), \((L', \cdot)\) be \(A\)-loops, \(f : L \to L'\) a half-isomorphism and \(x \in L\) such that the order of \(x\), denoted by \(o(x)\), is finite. Then \(o(f(x)) = o(x)\).
Lemma 1. Let $(L, \ast), (L', \cdot)$ be loops and $f : L \to L'$ a half-isomorphism. If $L'$ is commutative, then $L$ is commutative and $f$ is an isomorphism.

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Take $u \in G$ such that $o(u) > 2$. 
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f((0, u)) = (0, u') \quad f((0, uv)) = (0, w), \quad u', w \in G
\]

\[
(0, w) = f((0, uv)) = f((0, u) \ast (0, v)) \in \{(1, u'^{-1}v'), (1, v'u')\}
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$\Rightarrow o(v) = 2 \Rightarrow o(uv) > 2$

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Let \( f : \text{Dih}(\alpha, G) \rightarrow \text{Dih}(\beta, G) \) be a half-isomorphism. Define \( f' : G \rightarrow G \) by:

\[
(0, f'(u)) = f((0, u))
\]
Our Results

Let \( f : Dih(\alpha, G) \rightarrow Dih(\beta, G) \) be a half-isomorphism. Define \( f' : G \rightarrow G \) by:

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Proposition 3. The function \( f' \) defined above is an automorphism of \( G \).
Since \( f((0, v)) \in (0, G) \), \( \forall v \in G \), we have:
Since $f((0, v)) \in (0, G)$, $\forall v \in G$, we have:

$$f((1, e)) = (1, a), \text{ for some } a \in G$$
Since \( f((0, v)) \in (0, G), \forall v \in G \), we have:

\[
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\end{array}
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Thus:
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\end{align*}
\]

Thus:

\[
f((1,u)) = f((1,e) \ast (0,u)) \in \{(1,af'(u)), (1,af'(u^{-1}))\}
\]
Since $f((0, v)) \in (0, G)$, $\forall v \in G$, we have:

$$f((1, e)) = (1, a), \text{ for some } a \in G$$

Thus:

$$f((1, u)) = f((1, e) \ast (0, u)) \in \{(1, af'(u)), (1, af'(u^{-1}))\}$$

**Proposition 4.** Let $f : Dih(\alpha, G) \rightarrow Dih(\beta, G)$ be a half-isomorphism. Then there exists $a \in G$ such that for all $u \in G$, we have:

$$f((i, u)) = \begin{cases} 
(0, f'(u)), & i = 0 \\
(1, af'(u^{\epsilon_u})), & i = 1 \quad (\epsilon_u \in \{-1, 1\})
\end{cases}$$
Let $f' : G \rightarrow G$ be an automorphism and $a \in G$. Define $f_{-a}, f_{+a} : \text{Dih}(\alpha, G) \rightarrow \text{Dih}(\beta, G)$ by

$$f_{-a}((i, u)) = \begin{cases} (0, f'(u)), & i = 0 \\ (1, af'(u^{-1})), & i = 1 \end{cases}$$

$$f_{+a}((i, u)) = \begin{cases} (0, f'(u)), & i = 0 \\ (1, af'(u)), & i = 1 \end{cases}$$

Since $G$ does not have period 2, $f_{-a} \neq f_{+a}$, for all $a \in G$. 

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Let \( f' : G \rightarrow G \) be an automorphism and \( a \in G \). Define \( f_{-a}, f_{+a} : Dih(\alpha, G) \rightarrow Dih(\beta, G) \) by

\[
\begin{align*}
    f_{-a}((i, u)) &= \begin{cases} 
        (0, f'(u)), & i = 0 \\
        (1, af'(u^{-1})), & i = 1 
    \end{cases} \\
    f_{+a}((i, u)) &= \begin{cases} 
        (0, f'(u)), & i = 0 \\
        (1, af'(u)), & i = 1 
    \end{cases}
\end{align*}
\]

Since \( G \) does not have period 2, \( f_{-a} \neq f_{+a} \), for all \( a \in G \).
Proposition 5. Let $f' : G \to G$ be an automorphism and $a \in G$. Then:

a) If $f' \alpha = \beta f'$, then $f + a$ is an isomorphism and $f - a$ is an anti-isomorphism.

b) If $f' \alpha = \beta f' J$, then $f + a, f - a$ are non-trivial half-isomorphisms, where $J$ is the inversion map.

Proposition 6. Let $f : \text{Dih}(\alpha, G) \to \text{Dih}(\beta, G)$ be a half-isomorphism. If $f' \alpha \in \{\beta f', \beta f' J\}$, then $f \in \{f - a, f + a \mid a \in G\}$. 
Proposition 5. Let $f' : G \to G$ be an automorphism and $a \in G$. Then:

a) If $f'\alpha = \beta f'$, then $f_+a$ is an isomorphism and $f_-a$ is an anti-isomorphism.
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Proposition 6. Let $f : Dih(\alpha, G) \rightarrow Dih(\beta, G)$ be a half-isomorphism. If $f'\alpha \in \{ \beta f', \beta f'J \}$, then $f \in \{ f_{-a}, f_{+a} \mid a \in G \}$. 
The next theorem provides the answer of question 4.

Theorem. The statements are equivalent:

(a) \( f : \text{Dih}(\alpha, G) \rightarrow \text{Dih}(\beta, G) \) is a non trivial half-isomorphism.

(b) \( f \in \{ f - a, f + a \mid a \in G \} \), with \( f' : G \rightarrow G \) automorphism and \( f'\alpha = \beta f' \).

The sketch of proof:

(b) \(\Rightarrow\) (a) It follows by proposition 5.
Our Results

The next theorem provides the answer of question 4.

**Question 4.** What form do these non trivial half-isomorphisms take?
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**Theorem.** The statements are equivalent:

- \( f: \text{Dih}(\alpha,G) \to \text{Dih}(\beta,G) \) is a non trivial half-isomorphism.
- \( f \in \{f - a, f + a\} \) for some \( a \in G \), with \( f': \text{G} \to \text{G} \) automorphism and \( f'\alpha = \beta f' \).
Our Results

The next theorem provides the answer of question 4.

**Question 4.** What form do these non trivial half-isomorphisms take?

**Theorem.** The statements are equivalent:

a) $f : Dih(\alpha, G) \to Dih(\beta, G)$ is a non trivial half-isomorphism.
Our Results

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**Theorem.** The statements are equivalent:

a) \( f : Dih(\alpha, G) \rightarrow Dih(\beta, G) \) is a non trivial half-isomorphism.

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The sketch of proof:

(b) \( \Rightarrow \) (a) It follows by proposition 5.
Our Results

(a) => (b)
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\[ f((1, e) \ast (1, v)) = (0, f'\alpha(v)) \]
Our Results

(a) => (b)

\[ f((1, e) \ast (1, v)) = (0, f'\alpha(v)) \]

\[ f((1, e)) \cdot f((1, v)) = (0, \beta f'(v^{e_v})) \]
Our Results

(a) => (b)

\[ f((1, e) \ast (1, v)) = (0, f'\alpha(v)) \]
\[ f((1, e)) \cdot f((1, v)) = (0, \beta f'(v^e)) \]
\[ f((1, v)) \cdot f((1, e)) = (0, \beta f'(v^{-e})) \]
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\[ f'\alpha(v) \in \{ \beta f'(v), \beta f'(v^{-1}) \} \quad (\forall v \in G) \]
Our Results

(a) \implies (b)

\[ f((1, e) \ast (1, v)) = (0, f'\alpha(v)) \]

\[ f((1, e)) \cdot f((1, v)) = (0, \beta f'(v^e)) \]

\[ f((1, v)) \cdot f((1, e)) = (0, \beta f'(v^{-e})) \]

\[ f'\alpha(v) \in \{ \beta f'(v), \beta f'(v^{-1}) \} \quad (\forall v \in G) \]

\[ H = \{ v \in G \mid f'\alpha(v) = \beta f'(v) \} \leq G \]

\[ K = \{ v \in G \mid f'\alpha(v) = \beta f'J(v) \} \leq G \]
Our Results

(a) => (b)

\[ f((1, e) \ast (1, v)) = (0, f'\alpha(v)) \]
\[ f((1, e)) \cdot f((1, v)) = (0, \beta f'(v^e)) \]
\[ f((1, v)) \cdot f((1, e)) = (0, \beta f'(-e)v)) \]

\[ f'\alpha(v) \in \{\beta f'(v), \beta f'(v^{-1})\} \quad (\forall v \in G) \]

\[ H = \{v \in G \mid f'\alpha(v) = \beta f'(v)\} \leq G \]
\[ K = \{v \in G \mid f'\alpha(v) = \beta f'J(v)\} \leq G \]

=> \( G = H \cup K \)
Our Results

\[ \Rightarrow G \in \{H, K\} \]
Our Results

\[ G \in \{H, K\} \]

\[ f'\alpha \in \{\beta f', \beta f'J\} \]
Our Results

=> \( G \in \{H, K\} \)

=> \( f'\alpha \in \{\beta f', \beta f'J\} \)

Proposition 6

=> \( f \in \{f_-a, f_+a | a \in G\} \)
Our Results

$=> G \in \{H, K\}$

$=> f'\alpha \in \{\beta f', \beta f'J\}$

Proposition 6

$=> f \in \{f_a, f_+a | a \in G\}$

Since $f$ is a non trivial half-isomorphism, by Proposition 5 we have $f'\alpha = \beta f'J$. 
Our Results

\[ \Rightarrow G \in \{H, K\} \]

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Proposition 6

\[ \Rightarrow f \in \{f_{-a}, f_{+a} | a \in G\} \]

Since \( f \) is a non trivial half-isomorphism, by Proposition 5 we have \( f'\alpha = \beta f'J \).
The next corollary answers question 2.

**Question 2.** What are the conditions for the existence of a non trivial half-isomorphism of $Dih(\alpha, G)$ into $Dih(\beta, G)$?
The next corollary answers question 2.

Question 2. What are the conditions for the existence of a non trivial half-isomorphism of $\text{Dih}(\alpha, G)$ into $\text{Dih}(\beta, G)$?

Corollary 1. Let $G$ be a group such that period is not 2. Then there exists a non trivial half-isomorphism of $\text{Dih}(\alpha, G)$ into $\text{Dih}(\beta, G)$ if, and only if, $\alpha$ is conjugated to $\beta J$ in $\text{Aut}(G)$. 
The next corollary answers question 3.

**Question 3.** How many non trivial half-isomorphisms does exist of $Dih(\alpha, G)$ into $Dih(\beta, G)$?
Our Results

The next corollary answers question 3.

**Question 3.** How many non trivial half-isomorphisms does exist of \( Dih(\alpha, G) \) into \( Dih(\beta, G) \)?

**Corollary 2.** Let \( G \) be a group such that period is not 2 and \( \alpha, \beta \) automorphisms of \( G \) such that \( \alpha \) is conjugated to \( \beta J \). Then there are \( 2|G||C_{\text{Aut}(G)}(\alpha)| \) non trivial half-isomorphisms of \( Dih(\alpha, G) \) into \( Dih(\beta, G) \).

Where \( C_{\text{Aut}(G)}(\alpha) = \{ \psi \in \text{Aut}(G) \mid \alpha \psi = \psi \alpha \} \)
If $\alpha = I_d$, then $Dih(I_d, G)$ is the generalized dihedral group. Since $J^2 = I_d$, we have:
If $\alpha = I_d$, then $Dih(I_d, G)$ is the generalized dihedral group. Since $J^2 = I_d$, we have:

**Corollary 3.** Let $G$ be a group such that period is not 2. There are $2|G||Aut(G)|$ non trivial half-isomorphisms of $Dih(I_d, G)$ into $Dih(J, G)$. 
Consider $G = C_3$. Then $Dih(I_d, C_3)$ is the dihedral group of order 6 and $Dih(J, C_3)$ is the smallest non associative automorphic loop.

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Example

Consider $G = C_3$. Then $Dih(I_d, C_3)$ is the dihedral group of order 6 and $Dih(J, C_3)$ is the smallest non associative automorphic loop.
Example

The group $C_3 = \{1, 2, 3\}$ has two automorphisms: $I_d$ and $J$. 
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The group $C_3 = \{1, 2, 3\}$ has two automorphisms: $I_d$ and $J$. The non trivial half-isomorphisms between $Dih(I_d, C_3)$ and $Dih(J, C_3)$ are:

\[
\begin{align*}
I_{d-1} &= (5\ 6) & I_{d-2} &= (4\ 5) & I_{d-3} &= (4\ 6) \\
I_{d+1} &= I & I_{d+2} &= (4\ 5\ 6) & I_{d+3} &= (4\ 6\ 5)
\end{align*}
\]
The group $C_3 = \{1, 2, 3\}$ has two automorphisms: $I_d$ and $J$.

The non trivial half-isomorphisms between $Dih(I_d, C_3)$ and $Dih(J, C_3)$ are:

\[
\begin{align*}
I_d^{-1} & = (5 \ 6) & I_d^{-2} & = (4 \ 5) & I_d^{-3} & = (4 \ 6) \\
I_d^{+1} & = I & I_d^{+2} & = (4 \ 5 \ 6) & I_d^{+3} & = (4 \ 6 \ 5) \\
J^{-1} & = (2 \ 3) & J^{-2} & = (2 \ 3)(4 \ 5 \ 6) & J^{-3} & = (2 \ 3)(4 \ 6 \ 5) \\
J^{+1} & = (2 \ 3)(5 \ 6) & J^{+2} & = (2 \ 3)(4 \ 5) & J^{+3} & = (2 \ 3)(4 \ 6)
\end{align*}
\]
Thank you for your attention!
References


