Commutator theory for quandles

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Goal

... describe solvable / nilpotent quandles

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Corollaries:

- Topologically slice knots cannot be colored by latin quandles.
- Bruck loops of odd order are *really* solvable.

Solvability and nilpotence

A group G is solvable, resp. nilpotent, if there are $N_i \trianglelefteq G$ such that

$$1 = N_0 \leq N_1 \leq \ldots \leq N_k = G$$

and N_{i+1}/N_i is an abelian, resp. central subgroup of G/N_i, for all i.

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A general algebraic structure A is solvable, resp. nilpotent, if there are congruences α_i such that

$$0_A = \alpha_0 \leq \alpha_1 \leq ... \leq \alpha_k = 1_A$$

and α_{i+1}/α_i is an abelian, resp. central congruence of A/α_i, for all i.

Need a good notion of *abelianness* and *centrality* for congruences.

Solvability and nilpotence, via commutator

$$G^{(0)} = G_{(0)} = G, \qquad G_{(i+1)} = [G_{(i)}, G_{(i)}], \qquad G^{(i+1)} = [G^{(i)}, G]$$

A group G is

- solvable iff $G_{(n)} = 1$ for some n
- nilpotent iff $G^{(n)} = 1$ for some n

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$$\alpha^{(0)} = \alpha_{(0)} = 1_A, \qquad \alpha_{(i+1)} = [\alpha_{(i)}, \alpha_{(i)}], \qquad \alpha^{(i+1)} = [\alpha^{(i)}, 1_A]$$

A general algebraic structure A is

- solvable iff $\alpha_{(n)} = 0_A$ for some n
- nilpotent iff $\alpha^{(n)} = 0_A$ for some n

Need a good notion of *commutator of congruences*.

Commutator theory

[mid 1970s by Smith, Gumm, Herrmann, ..., the Freese-McKenzie 1987 book]

Centralizing relation for congruences α, β, δ of A:

$C(\alpha, \beta; \delta)$ iff for every term $t(x, y_1, \dots, y_n)$ and every $a \stackrel{\alpha}{\equiv} b$, $u_i \stackrel{\beta}{\equiv} v_i$

$$t(a, u_1, \dots, u_n) \stackrel{\delta}{\equiv} t(a, v_1, \dots, v_n) \;\Rightarrow\; t(b, u_1, \dots, u_n) \stackrel{\delta}{\equiv} t(b, v_1, \dots, v_n)$$

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The *commutator* $[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$.

A congruence α is called

- abelian if $C(\alpha, \alpha; 0_A)$, i.e., if $[\alpha, \alpha] = 0_A$.
- central if $C(\alpha, 1_A; 0_A)$, i.e., if $[\alpha, 1_A] = 0_A$.

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An easy but non-trivial fact:
In groups, this gives the usual commutator, abelianness, centrality.

A deep theory: works well in varieties with modular congruence lattices.

Abelian algebras

An algebra A is called abelian if 1_A is abelian, i.e., if $[1_A, 1_A] = 0_A$, i.e., if

$$t(a, u_1, \dots, u_n) = t(a, v_1, \dots, v_n) \ \Rightarrow \ t(b, u_1, \dots, u_n) = t(b, v_1, \dots, v_n)$$

for every term $t(x, y_1, \dots, y_n)$ and every a, b, u_i, v_i.

Observation

Modules are abelian.

Proof: $t(x, y_1, \dots, y_n) = rx + \sum r_i y_i$, cancel ra, add rb.

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Observation

An abelian group is commutative.

Proof: $t(x, y, z) = yxz, \quad a11 = 11a \;\Rightarrow\; ab1 = 1ba$

Observation

An abelian loop is a commutative group.

Pf: $t = (xy)(uv), \quad (11)(bc) = (1b)(1c) \;\Rightarrow\; (a1)(bc) = (ab)(1c)$

Abelian algebras = modules

Observation

Modules are abelian.

Theorem (Gumm-Smith, 1970s)

In a variety with modular congruence lattices, TFAE

1. *A is abelian*
2. *A is polynomially equivalent to a module*

Example: groups, loops, quasigroups

Non-example: quandles

Quandles

An algebraic structure $(Q, *, \backslash)$ is called a *quandle* if

- $x * x = x$
- all left translations $L_x(y) = x * y$ are automorphisms, with $L_x^{-1}(y) = x \backslash y$.

Multiplication group, displacement group:

$$\mathrm{LMlt}(Q) = \langle L_x : x \in Q \rangle \leq \mathrm{Aut}(Q)$$
$$\mathrm{Dis}(Q) = \langle L_x L_y^{-1} : x, y \in Q \rangle \leq \mathrm{LMlt}(Q)$$

A quandle is called *connected* if $\mathrm{LMlt}(Q)$ is transitive on Q.

Affine quandles (aka Alexander) $Aff(A, f)$:
$x * y = (1 - f)(x) + f(y)$ on an abelian group A, $f \in Aut(A)$

Abelian quandles

Theorem (Jedlička, Pilitowska, S., Zamojska-Dzienio)

TFAE for a quandle Q:

1. *abelian*
2. *embeds into (a reduct of) a module*
3. $\mathrm{Dis}(Q)$ *abelian, semiregular*
4. $Q \simeq Ext(A, f, \bar{d})$, *a certain kind of extension of* $Aff(A, f)$

... see Přemysl's talk for details (and much more)

Congruences of quandles

Let $N(Q) = \{N \leq \mathrm{Dis}(Q) : N \text{ is normal in } \mathrm{LMlt}(Q)\}$

There is a Galois correspondence

$$Con(Q) \longleftrightarrow N(Q)$$

$$\alpha \to \mathrm{Dis}_\alpha(Q) = \langle L_x L_y^{-1} : x \,\alpha\, y \rangle$$

$$\alpha_N = \{(x, y) : L_x L_y^{-1} \in N\} \leftarrow N$$

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Proposition (Bonatto, S.)

TFAE for $\alpha, \beta \in Con(Q)$, Q a quandle:

1. *α centralizes β over 0_Q, i.e., $C(\alpha, \beta; 0_Q)$*
2. *$\mathrm{Dis}_\beta(Q)$ centralizes $\mathrm{Dis}_\alpha(Q)$ and acts semiregularly on every α-block*

Abelian congruences and solvable quandles

Theorem

TFAE for a congruence α of a quandle Q:

1. *α is abelian*
2. *$\mathrm{Dis}_\alpha(Q)$ is abelian and acts semiregularly on every block of α*
3. *Q is an abelian extension of $F = Q/\alpha$, i.e., $(F \times A, *)$ with*
$$(x, a) * (y, b) = (xy,\ \varphi_{x,y}(a) + \psi_{x,y}(b) + \theta_{x,y})$$
where A is an abelian group, $\varphi : Q^2 \to End(A)$, $\psi : Q^2 \to Aut(A)$, $\theta : Q^2 \to A$ satisfying the cocycle condition.

The last item only assuming that α has connected blocks.

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The last item only assuming that α *has connected blocks.*

Corollary

- Q *solvable (of rank* n*)* $\Rightarrow \mathrm{Dis}(Q)$ *solvable (of rank* $\leq 2n-1$*)*
- $\mathrm{Dis}(Q)$ *solvable,* Q *is superconnected* $\Rightarrow Q$ *solvable*

superconnected = all subquandles are conected (example: latin quandles)

Central congruences and nilpotent quandles

Theorem

TFAE for a congruence α of a quandle Q:

1. α *is central*
2. $\mathrm{Dis}_\alpha(Q)$ *is central and* $\mathrm{Dis}(Q)$ *acts semiregularly on every block of* α
3. Q *is a central extension of* $F = Q/A$, *i.e.,* $(F \times A, *)$ *with*
$$(x, a) * (y, b) = (xy,\ (1 - f)(a) + f(b) + \theta_{x,y})$$
where A *is an abelian group,* $\theta : Q^2 \to A$ *satisfying the cocycle condition.*

The last item only assuming that Q is superconnected.

Corollary

- Q *nilpotent (of rank* n*)* $\Rightarrow \mathrm{Dis}(Q)$ *nilpotent (of rank* $\leq 2n - 1$*)*
- $\mathrm{Dis}(Q)$ *nilpotent,* Q *is superconnected* $\Rightarrow Q$ *nilpotent*

Extensions by constant cocycles (aka coverings)

Theorem

TFAE for a congruence α of a quandle Q:

1. α *is strongly abelian*
2. $\mathrm{Dis}_\alpha(Q) = 1$
3. Q *is an extension by constant cocycle of* $F = Q/\alpha$*, i.e.,* $(F \times A, *)$ *with*
   $$(x, a) * (y, b) = (xy,\ \rho_{x,y}(b))$$
   where A *is a set,* $\rho : Q^2 \to Sym(A)$ *satisfying the cocycle condition.*

... coverings are a special case of our abelian extensions ($\varphi_{x,y} = 0$)

... coverings have a natural universal algebraic meaning (*strongly abelian congruences*)

Abeliannes for quandles vs. loops

Theorem

TFAE for a congruence α of a quandle Q:

1. α *is abelian*
2. $\mathrm{Dis}_\alpha(Q)$ *is abelian and acts semiregularly on every block of* α
3. Q *is an abelian extension of* $F = Q/\alpha$*, i.e.,* $(F \times A, *)$ *with*

$$(x, a) * (y, b) = (xy,\ \varphi_{x,y}(a) + \psi_{x,y}(b) + \theta_{x,y})$$

Theorem (S., Vojtěchovský)

TFAE for a normal subloop $A \trianglelefteq Q$ of a loop:

1. A *is abelian (in* Q*)*
2. $\varphi_{r,s}(a) = \varphi_{u,v}(a)$ *for every* $a, r/u, s/v \in A$, $\varphi \in \{L, R, T\} \subseteq Inn(Q)$
3. Q *is an abelian extension of* $F = Q/A$*, i.e.,* $(F \times A, *)$ *with*

$$(x, a) * (y, b) = (xy,\ \varphi_{x,y}(a) + \psi_{x,y}(b) + \theta_{x,y})$$

Centrality for quandles vs. loops

Theorem

TFAE for a congruence α of a quandle Q:

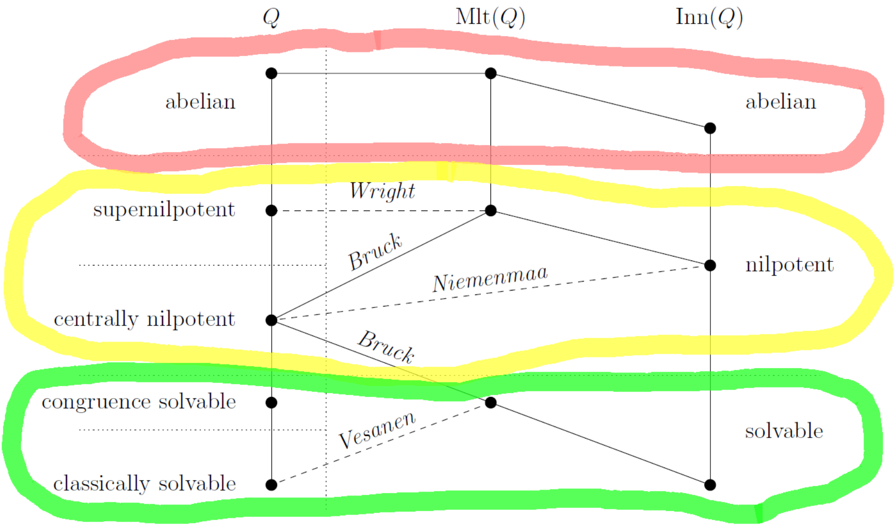
1. α *is central*
2. $\mathrm{Dis}_\alpha(Q)$ *is central and* $\mathrm{Dis}(Q)$ *acts semiregularly on every block of* α
3. Q *is a central extension of* $F = Q/A$, *i.e.,* $(F \times A, *)$ *with*
$$(x, a) * (y, b) = (xy,\ (1 - f)(a) + f(b) + \theta_{x,y})$$

Theorem

TFAE for a normal subloop $A \trianglelefteq Q$ of a loop:

1. A *is central (in* Q*)*
2. $\varphi_{r,s}(a) = a$ *for every* $a \in A$, $r, s \in Q$, $\varphi \in \{L, R, T\} \subseteq Inn(Q)$
3. Q *is an central extension of* $F = Q/A$, *i.e.,* $(F \times A, *)$ *with*
$$(x, a) * (y, b) = (xy,\ a + b + \theta_{x,y})$$

Solvability and nilpotence for loops



An application to quandles

Classification of latin quandles of order $3p$. See Marco's talk.

An application to loop theory

Theorem (Stein 2001)

If Q is a finite latin quandle, then $\mathrm{LMlt}(Q)$ is solvable.

Since latin quandles are superconnected, we obtain

Corollary

Finite latin quandles are solvable.

In particular,

- involutory latin quandles are solvable,
- all of their polynomial reducts are solvable,
- in particular,

Corollary

Bruck loops of odd order are solvable (in the stronger sense).

An application to knot theory

Coloring by affine quandles $\leftrightsquigarrow$ *Alexander invariant*

Theorem (Bae, 2011)

Let K be a link and f its Alexander polynomial.

- $f = 0 \Rightarrow$ *colorable by every affine quandle*
- $f = 1 \Rightarrow$ *not colorable by any affine quandle*
- *else, colorable by* $\mathrm{Aff}(\mathbb{Z}[t, t^{-1}]/(f), f)$.

Corollary

- $f = 1 \Rightarrow$ *not colorable by any solvable quandle (in particular, latin)*