Homotopy link invariants from geometric realizations of SD-homology

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July 31, 2017
Fourth Mile High Conference on Nonassociative Mathematics

Contents

Geometric realizations of SD-homology

Definition

A **pre-simplicial set** $\mathcal{X} = (X_n, d_i)$ consists of a collection of sets X_n for $n \geqslant 0$ and face maps $d_i := d_{i,n} : X_n \to X_{n-1}$ for $0 \leqslant i \leqslant n$ satisfying $d_i d_j = d_{j-1} d_i$ if $i < j$.

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For a commutative ring k with unity, we let $C_n = kX_n$ and $\partial_n = \sum_{i=0}^{n} (-1)^i d_i$. Then (C_n, ∂_n) forms a chain complex, so we can define homology groups of it.

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The **geometric realization** $|\mathcal{X}|$ of a pre-simplicial set $\mathcal{X}$ is a CW-complex defined as the quotient of the disjoint union $\coprod_n (X_n \times \Delta^n)$ by $(d_i(\mathbf{x}), \mathbf{t}) \sim (\mathbf{x}, d^i(\mathbf{t}))$, where Δ^n be the standard n-simplex and $d^i := d^{i,n} : \Delta^{n-1} \to \Delta^n$ are maps defined by $d^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$ for $0 \leqslant i \leqslant n$.

Definition

A **pre-cubic set** $\mathcal{X} = (X_n, d_i^{\varepsilon})$ consists of a collection of sets X_n for $n \geqslant 0$ and face maps $d_i^{\varepsilon} := d_{i,n}^{\varepsilon} : X_n \to X_{n-1}$ for $1 \leqslant i \leqslant n$ and $\varepsilon \in \{0, 1\}$ satisfying $d_i^{\varepsilon} d_j^{\delta} = d_{j-1}^{\delta} d_i^{\varepsilon}$ for $i < j$ and $\delta, \varepsilon \in \{0, 1\}$.

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SD-homology

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Definition

A **right SD structure** (or a **shelf**) $(X, *)$ is a set X with the right self-distributive binary operation $* : X \times X \to X$ (i.e. $(a * b) * c = (a * c) * (b * c)$ for all $a, b, c \in X$).

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Definition (Przytycki, 2014)

The homology obtained from the pre-simplicial set $(X^{n+1}, d_i^{(*)})$ (the pre-cubic set $(X^{n+1}, d_i^{(*)}, d_i^{(*_0)})$, respectively) is said to be a **one-term** (**two-term**, respectively) **SD-homology**.

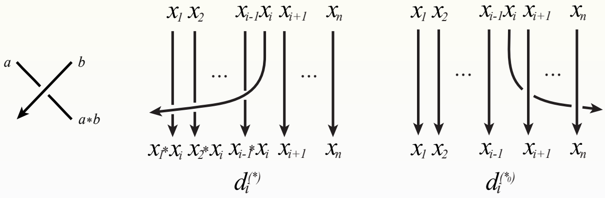

Figure: Graphical descriptions of face maps $d_i^{(*)}$ and $d_i^{(*_0)}$.

Now, we consider a special RDS motivated by Knot Theory.

Theorem (Reidemeister/Alexander and Briggs)

Let D_1 and D_2 be diagrams of classical knots K_1 and K_2, respectively. Then K_1 and K_2 are equivalent if and only if D_1 can be deformed to D_2 by a finite sequence of Reidemeister moves.

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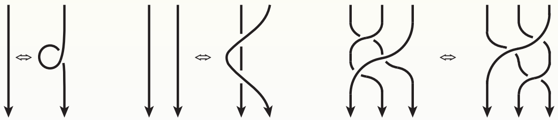

Figure: Quandle axioms from Reidemeister moves.

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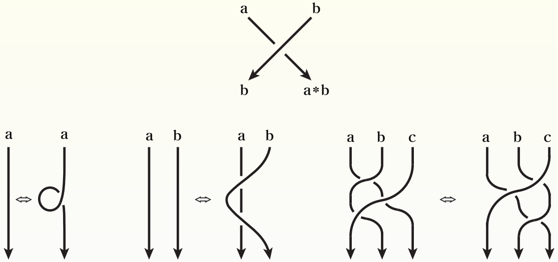

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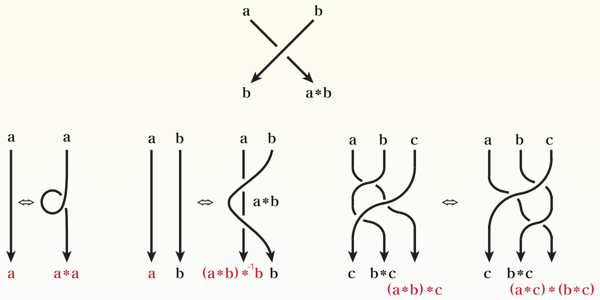

Figure: Quandle axioms from Reidemeister moves.

Quandles

Definition (Joyce/Matveev, 1982)

A **quandle** $(X, *)$ is an algebraic structure with a set X and a binary operation $* : X \times X \to X$ satisfying the following axioms:

1. (Idempotency) For any $a \in X$, $a * a = a$.
2. (Invertibility) For each $b \in X$, $*_b : X \to X$ given by $*_b(x) = x * b$ is invertible.
3. (Right self-distributivity) For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Notice that the three quandle axioms above are motivated by Knot Theory.

Rack Homology Groups and Rack Spaces

Definition (Fenn, Rourke, Sanderson, 1993)

For a rack X, the integral homology obtained from the pre-cubic set $(X^n, d_i^{(*)}, d_i^{(*_0)})$ (or $(X^{n+1}, d_{i+1}^{(*)}, d_{i+1}^{(*_0)})$) is called the **rack homology** of X.

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$$\text{i.e. } C_n(X) = \mathbb{Z}X^n \text{ and } \partial_n = \sum_{i=1}^{n} (-1)^i (d_i^{(*)} - d_i^{(*_0)})$$

$$\text{(or } C_n(X) = \mathbb{Z}X^{n+1} \text{ and } \partial_n = \sum_{i=2}^{n} (-1)^i (d_{i+1}^{(*)} - d_{i+1}^{(*_0)})\text{).}$$

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The geometric realization of the pre-cubic set $(X^n, d_i^{(*)}, d_i^{(*_0)})$ ($(X^{n+1}, d_{i+1}^{(*)}, d_{i+1}^{(*_0)})$, respectively) is said to be the **rack space** (**extended rack space**, respectively) of X.
We denote it by BX ($B_X X$, respectively).

Quandle Homology

For a quandle X, we consider the subgroup $C_n^D(X)$ of $C_n(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ of elements of X with $x_i = x_{i+1}$ for some $i = 1, \ldots, n-1$.
Notice that $(C_n^D(X), \partial_n)$ is a subchain complex of a rack chain complex $(C_n(X), \partial_n)$.

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Definition (Carter, Jelsovsky, Kamada, Langford, Saito, 2001)

For a quandle X, the quotient chain complex $(C_n^Q(X), \partial_n) = (C_n(X)/C_n^D(X), \partial_n)$ is called the **quandle chain complex** of X.

An application to Knot Theory

-Shadow homotopy invariants of classical links-

Rack Space and Extended Rack Space

Definition (Fenn, Rourke, Sanderson, 1993 / 1995)

Let X be a rack and let $d_i^{(*_0)}, d_i^{(*)}$ be face maps in the boundary homomorphism of rack homology.
The geometric realization of the pre-cubic set $(X^n, d_i^{(*_0)}, d_i^{(*)})$ is called the **rack space** BX of X.

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The geometric realization of the pre-cubic set $(X^n, d_i^{(*_0)}, d_i^{(*)})$ is called the **rack space** BX of X.
Especially, the geometric realization of $(X^{n+1}, d_{i+1}^{(*_0)}, d_{i+1}^{(*)})$ is said to be the **extended rack space** $B_X X$ of X.

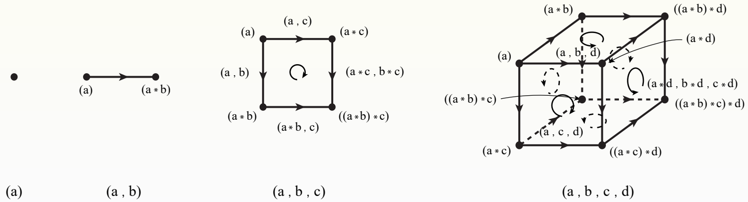

Figure: Low-dimensional cells of an extended rack space.

Extended Quandle Space and Action Quandle Space

For a quandle X, Nosaka introduced the quandle space $B^Q X$ modifying the rack space of X, and defined the quandle homotopy invariant using quandle spaces.

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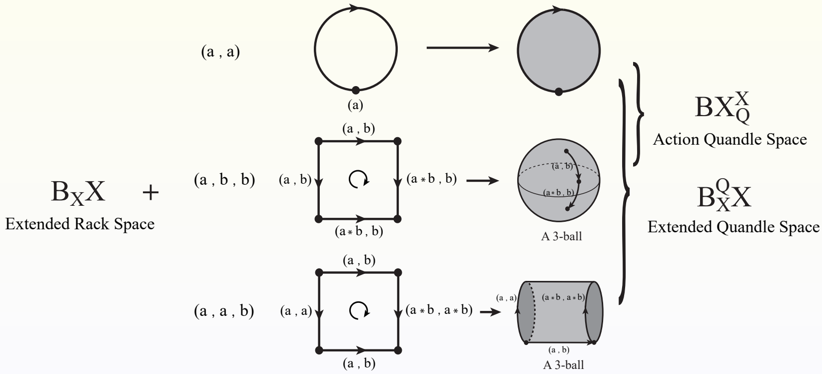

The shadow homotopy invariant of classical links

[Ingredients]

- An oriented link diagram D on I^2.
- A shadow coloring $\widetilde{\mathcal{C}}$ of D by a quandle X.
- The extended quandle space $B^Q_X X$ (or the action quandle space BX^X_Q) of X.

The shadow homotopy invariant of classical links

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- An oriented link diagram D on I^2.
- A shadow coloring $\widetilde{\mathcal{C}}$ of D by a quandle X.
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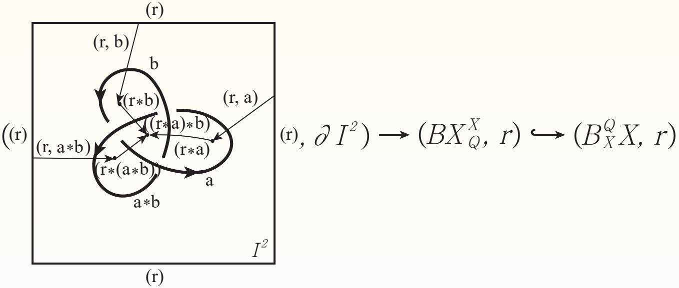

Figure: A shadow homotopy invariant of an oriented knot.

The shadow homotopy invariant of classical links

Theorem (Y., 2017)

Let $\psi_X(D_L; \widetilde{\mathcal{C}}) : (I^2, \partial I^2) \to (B_X^Q X, r) (or\ (BX_Q^X, r))$ *be the map defined as above. We denote by* $\Psi_X(L; \widetilde{\mathcal{C}})$ *the homotopy class of* $\psi_X(D_L; \widetilde{\mathcal{C}})$ *in* $\pi_2(B_X^Q X)$ *(or* $\pi_2(BX_Q^X)$*).*
Then $\Psi_X(L; \widetilde{\mathcal{C}})$ *is invariant under Reidemeister moves.*

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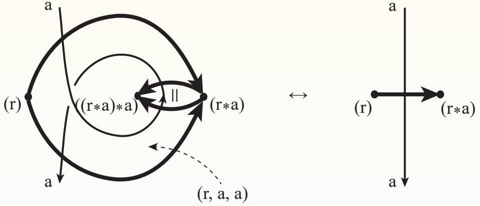
Then $\Psi_X(L; \widetilde{\mathcal{C}})$ is invariant under Reidemeister moves.

Proof



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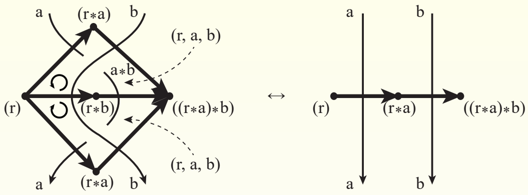

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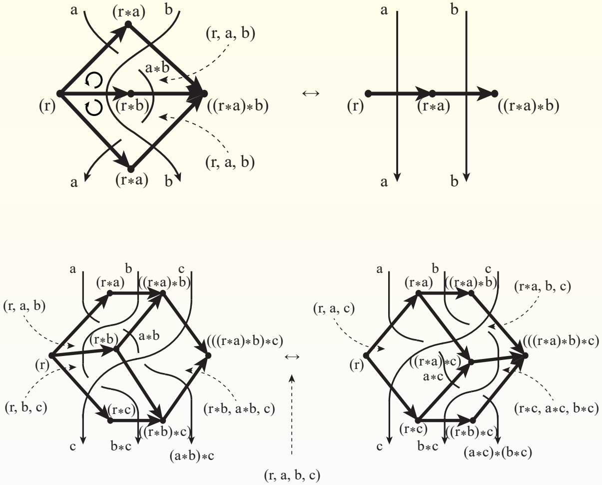


The shadow homotopy invariant of classical links

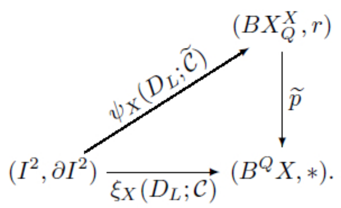
Definition (Y., 2017)

For a connected quandle X, if we let

$$\Psi_X(L) = \sum_{\widetilde{\mathcal{C}} \in SCol_X(L)} \Psi_X(L; \widetilde{\mathcal{C}}) \in \mathbb{Z}[\pi_2(B_X^Q X)] (\text{or } \mathbb{Z}[\pi_2(BX_Q^X)]),$$

then $\Psi_X(L)$ is a link invariant called the **shadow homotopy invariant** of an oriented link L.

The shadow homotopy invariant of classical links



Theorem (Y., 2017)

Let X be a finite connected quandle, and let $\Xi_X(L)$ be the quandle homotopy invariant of an oriented link L. Then

$$\Psi_X(L) = |X|\Xi_X(L).$$

Thank you for your attention!