Homotopy link invariants from geometric realizations of SD-homology

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1. Geometric realizations of SD-homology

2. An application to Knot Theory
   - Shadow homotopy invariants of classical links -
Geometric realizations of SD-homology

An application to Knot Theory: shadow homotopy invariants
Definition

A **pre-simplicial set** $\chi = (X_n, d_i)$ consists of a collection of sets $X_n$ for $n \geq 0$ and face maps $d_i := d_{i,n} : X_n \to X_{n-1}$ for $0 \leq i \leq n$ satisfying $d_id_j = d_{j-1}d_i$ if $i < j$. 
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For a commutative ring $k$ with unity, we let $C_n = kX_n$ and $\partial_n = \sum_{i=0}^{n} (-1)^i d_i$. Then $(C_n, \partial_n)$ forms a chain complex, so we can define homology groups of it.
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Definition
The **geometric realization** $|\chi|$ of a pre-simplicial set $\chi$ is a CW-complex defined as the quotient of the disjoint union $\bigsqcup_{n} (X_n \times \Delta^n)$ by $(d_i(x), t) \sim (x, d^i(t))$, where $\Delta^n$ be the standard $n$-simplex and $d^i := d^{i,n} : \Delta^{n-1} \rightarrow \Delta^n$ are maps defined by $d^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$ for $0 \leq i \leq n$. 
Definition

A pre-cubic set $\chi = (X_n, d_i^\varepsilon)$ consists of a collection of sets $X_n$ for $n \geq 0$ and face maps $d_i^\varepsilon := d_i^\varepsilon : X_n \to X_{n-1}$ for $1 \leq i \leq n$ and $\varepsilon \in \{0, 1\}$ satisfying $d_i^\varepsilon d_j^\delta = d_j^{\delta-1} d_i^\varepsilon$ for $i < j$ and $\delta, \varepsilon \in \{0, 1\}$. 

For a commutative ring $k$ with unity, we let $C_n = k X_n$ and $\partial_n = \sum_{i=1}^n (-1)^i (d_i^0 - d_i^1)$. Then $(C_n, \partial_n)$ forms a chain complex, so we can define homology groups of it.

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An application to Knot Theory: shadow homotopy invariants

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\coprod_n (X_n \times I^n) \text{ by } (d_i^\varepsilon(x), t) \sim (x, d_i^\varepsilon(t)), \text{ where } I^n = [0, 1]^n \subset \mathbb{R}^n
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\((n \geq 0) \) and \( d_i^\varepsilon := d_{i,n}^\varepsilon : I^{n-1} \to I^n \) are maps defined by
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\]
SD-homology

Definition

A right SD structure (or a shelf) \((X, \ast)\) is a set \(X\) with the right self-distributive binary operation \(\ast: X \times X \rightarrow X\) (i.e. \((a \ast b) \ast c = (a \ast c) \ast (b \ast c)\) for all \(a, b, c \in X\)).

Definition (Przytycki, 2014)

The homology obtained from the pre-simplicial set \((X_n + 1, d_\ast)\) (the pre-cubic set \((X_n + 1, d_\ast, d_0)\), respectively) is said to be a one-term (two-term, respectively) SD-homology.
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The homology obtained from the pre-simplicial set $(X^{n+1}, d_i^{(\star)})$ (the pre-cubic set $(X^{n+1}, d_i^{(\star)}, d_i^{(*0)})$, respectively) is said to be a **one-term** (**two-term**, respectively) **SD-homology**.

![Graphical descriptions of face maps $d_i^{(\star)}$ and $d_i^{(*0)}$.](image-url)
Now, we consider a special RDS motivated by Knot Theory.

**Theorem (Reidemeister/Alexander and Briggs)**

Let $D_1$ and $D_2$ be diagrams of classical knots $K_1$ and $K_2$, respectively. Then $K_1$ and $K_2$ are equivalent if and only if $D_1$ can be deformed to $D_2$ by a finite sequence of Reidemeister moves.
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**Figure:** Quandle axioms from Reidemeister moves.
Quandles

Definition (Joyce/Matveev, 1982)

A **quandle** \((X, \star)\) is an algebraic structure with a set \(X\) and a binary operation \(\star : X \times X \to X\) satisfying the following axioms:

1. (Idempotency) For any \(a \in X\), \(a \star a = a\).
2. (Invertibility) For each \(b \in X\), \(\star_b : X \to X\) given by \(\star_b(x) = x \star b\) is invertible.
3. (Right self-distributivity) For any \(a, b, c \in X\),
   \[(a \star b) \star c = (a \star c) \star (b \star c).\]

Notice that the three quandle axioms above are motivated by Knot Theory.
Rack Homology Groups and Rack Spaces

Definition (Fenn, Rourke, Sanderson, 1993)

For a rack $X$, the integral homology obtained from the pre-cubic set $(X^n, d_i^{(\ast)}, d_i^{(\ast_0)})$ (or $(X^{n+1}, d_{i+1}^{(\ast)}, d_{i+1}^{(\ast_0)})$) is called the rack homology of $X$. 

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The geometric realization of the pre-cubic set $(X^n, d_i^{(*)}, d_i^{(*0)})$ ($(X^{n+1}, d_{i+1}^{(*)}, d_{i+1}^{(*0)})$, respectively) is said to be the rack space (extended rack space, respectively) of $X$. We denote it by $BX$ ($B_X X$, respectively).
Quandle Homology

For a quandle $X$, we consider the subgroup $C^D_n(X)$ of $C_n(X)$ generated by $n$-tuples $(x_1, \ldots, x_n)$ of elements of $X$ with $x_i = x_{i+1}$ for some $i = 1, \ldots, n-1$.

Notice that $(C^D_n(X), \partial_n)$ is a subchain complex of a rack chain complex $(C_n(X), \partial_n)$. 

**Definition (Carter, Jelsovsky, Kamada, Langford, Saito, 2001)**

For a quandle $X$, the quotient chain complex $(C_Q^n(X), \partial_n) = (C_n(X)/C^D_n(X), \partial_n)$ is called the quandle chain complex of $X$. 

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**Geometric realizations of SD-homology**

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**An application to Knot Theory: shadow homotopy invariants**
Quandle Homology

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Notice that $(C_n^D(X), \partial_n)$ is a subchain complex of a rack chain complex $(C_n(X), \partial_n)$.

Definition (Carter, Jelsovsky, Kamada, Langford, Saito, 2001)

For a quandle $X$, the quotient chain complex $(C_n^Q(X), \partial_n) = (C_n(X)/C_n^D(X), \partial_n)$ is called the quandle chain complex of $X$. 
An application to Knot Theory

-Shadow homotopy invariants of classical links-
Rack Space and Extended Rack Space

Definition (Fenn, Rourke, Sanderson, 1993 / 1995)

Let $X$ be a rack and let $d_i^{(0)}$, $d_i^{(*)}$ be face maps in the boundary homomorphism of rack homology. The geometric realization of the pre-cubic set $(X^n, d_i^{(0)}, d_i^{(*)})$ is called the rack space $BX$ of $X$. 

Figure: Low-dimensional cells of an extended rack space.
Rack Space and Extended Rack Space

Definition (Fenn, Rourke, Sanderson, 1993 / 1995)
Let $X$ be a rack and let $d_i^{(*_0)}$, $d_i^{(*)}$ be face maps in the boundary homomorphism of rack homology. The geometric realization of the pre-cubic set $(X^n, d_i^{(*_0)}, d_i^{(*)})$ is called the rack space $BX$ of $X$. Especially, the geometric realization of $(X^{n+1}, d_{i+1}^{(*_0)}, d_{i+1}^{(*)})$ is said to be the extended rack space $B_XX$ of $X$.

Figure: Low-dimensional cells of an extended rack space.
Extended Quandle Space and Action Quandle Space

For a quandle $X$, Nosaka introduced the quandle space $B^Q X$ modifying the rack space of $X$, and defined the quandle homotopy invariant using quandle spaces.
Extended Quandle Space and Action Quandle Space

For a quandle $X$, Nosaka introduced the quandle space $B^Q_X$ modifying the rack space of $X$, and defined the quandle homotopy invariant using quandle spaces.

**Definition**

\[
B^X_X + \quad B^Q_X
\]

Extended Rack Space

Action Quandle Space

Extended Quandle Space
The shadow homotopy invariant of classical links

[Ingredients]

- An oriented link diagram $D$ on $I^2$.
- A shadow coloring $\widetilde{C}$ of $D$ by a quandle $X$.
- The extended quandle space $B^Q_X(X)$ (or the action quandle space $BX^X_Q$) of $X$. 
The shadow homotopy invariant of classical links

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- An oriented link diagram $D$ on $I^2$.
- A shadow coloring $\tilde{C}$ of $D$ by a quandle $X$.
- The extended quandle space $B_X^Q X$ (or the action quandle space $B_{X_X}^Q$) of $X$.

Figure: A shadow homotopy invariant of an oriented knot.
The shadow homotopy invariant of classical links

Theorem (Y., 2017)

Let \( \psi_X(D_L; \tilde{C}) : (I^2, \partial I^2) \to (B^Q_X X, r) \) (or \( (B^X_Q X, r) \)) be the map defined as above. We denote by \( \Psi_X(L; \tilde{C}) \) the homotopy class of \( \psi_X(D_L; \tilde{C}) \) in \( \pi_2(B^Q_X X) \) (or \( \pi_2(B^X_Q X) \)).

Then \( \Psi_X(L; \tilde{C}) \) is invariant under Reidemeister moves.
The shadow homotopy invariant of classical links

Theorem (Y., 2017)

Let $\psi_X(D_L; \tilde{C}) : (I^2, \partial I^2) \to (B^Q_X, r) \text{ (or } B^X_Q, r) \text{) be the map defined as above. We denote by $\Psi_X(L; \tilde{C})$ the homotopy class of $\psi_X(D_L; \tilde{C})$ in $\pi_2(B^Q_X, X) \text{ (or } \pi_2(B^X_Q, X) \text{).}$

Then $\Psi_X(L; \tilde{C})$ is invariant under Reidemeister moves.

Proof
The shadow homotopy invariant of classical links
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Definition (Y., 2017)

For a connected quandle $X$, if we let

$$
\Psi_X(L) = \sum_{\tilde{C} \in SCol_X(L)} \Psi_X(L; \tilde{C}) \in \mathbb{Z}[\pi_2(B^Q_X X)] \text{ (or } \mathbb{Z}[\pi_2(B^X_X Q)])
$$

then $\Psi_X(L)$ is a link invariant called the **shadow homotopy invariant** of an oriented link $L$. 
The shadow homotopy invariant of classical links

**Theorem (Y., 2017)**

Let $X$ be a finite connected quandle, and let $\Xi_X(L)$ be the quandle homotopy invariant of an oriented link $L$. Then

$$\Psi_X(L) = |X| \Xi_X(L).$$
Thank you for your attention!