MATH 1951 Exam 1

Name: Solutions

Instructions: This test should have 6 pages and 6 problems, and is out of 100 points. Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this exam. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. (a) (12 pts.) For the function $f(x) = x^2 - x$, find the derivative $f'(x)$ by using THE LIMIT DEFINITION OF THE DERIVATIVE.

$$f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x} - h - \cancel{x^2} + \cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 1)}{\cancel{h}} = \lim_{h\to 0} 2x + h - 1 = \boxed{2x - 1}$$

(b) (4 pts.) Use your answer to part (a) to find the equation of the tangent line to $f(x)$ at the point $(3, 6)$.

At $x = 3$:

$m = f'(3) = 2 \cdot 3 - 1 = 5$

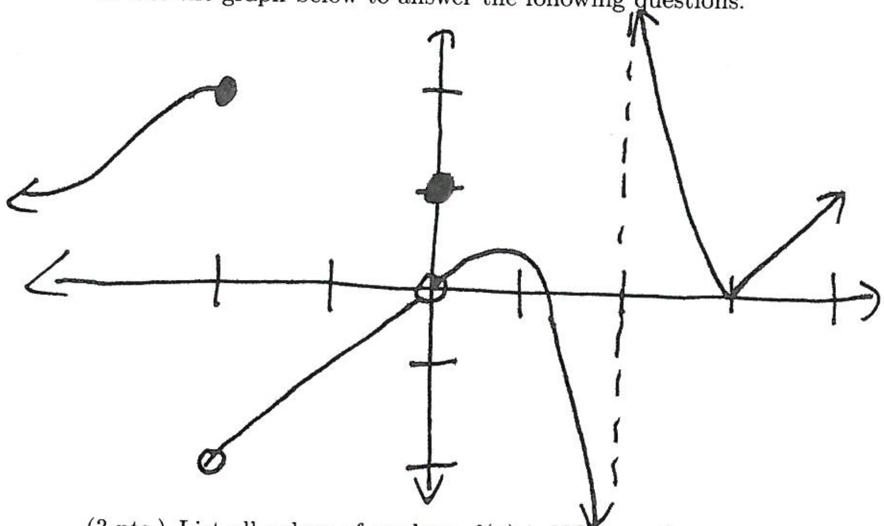
Point-slope:

$y - 6 = 5(x - 3)$

$y - 6 = 5x - 15$

$\boxed{y = 5x - 9}$

2. Use the graph below to answer the following questions.



(3 pts.) List all values of x where $f(x)$ is NOT continuous.

$x = -2, 0, 2$

(3 pts.) List all values of x where $f(x)$ is NOT differentiable.

$x = -2, 0, 2, 3$

(3 pts. each) Find the following limits: (you do not need to show work for these!)

$\lim_{x\to 2^+} f(x)$ $= \infty$

$\lim_{x\to 2^-} f(x)$ $= -\infty$

$\lim_{x\to 2} f(x)$ $=$ DNE

$\lim_{x\to -1}(f(x))^2$

$\lim_{x\to -1^+} f(x)^2 = (-1)^2 = 1$

3. (15 pts.) Find all horizontal and vertical asymptotes of $f(x) = \frac{2x^2+7x}{x^2-9}$. Use your answers to decide which of the graphs below is a graph of $y = f(x)$. (HINT: one-sided limits at the vertical asymptotes!)

$$\text{HA: } \lim_{x\to\pm\infty} \frac{2x^2+7x}{x^2-9} = \lim_{x\to\pm\infty} \frac{\frac{2x^2}{x^2} + \cancel{\frac{7x}{x^2}} \to 0}{\frac{x^2}{x^2} - \cancel{\frac{9}{x^2}} \to 0}$$

$$= \frac{2}{1} = 2 \qquad \text{HA at } \boxed{y=2}$$

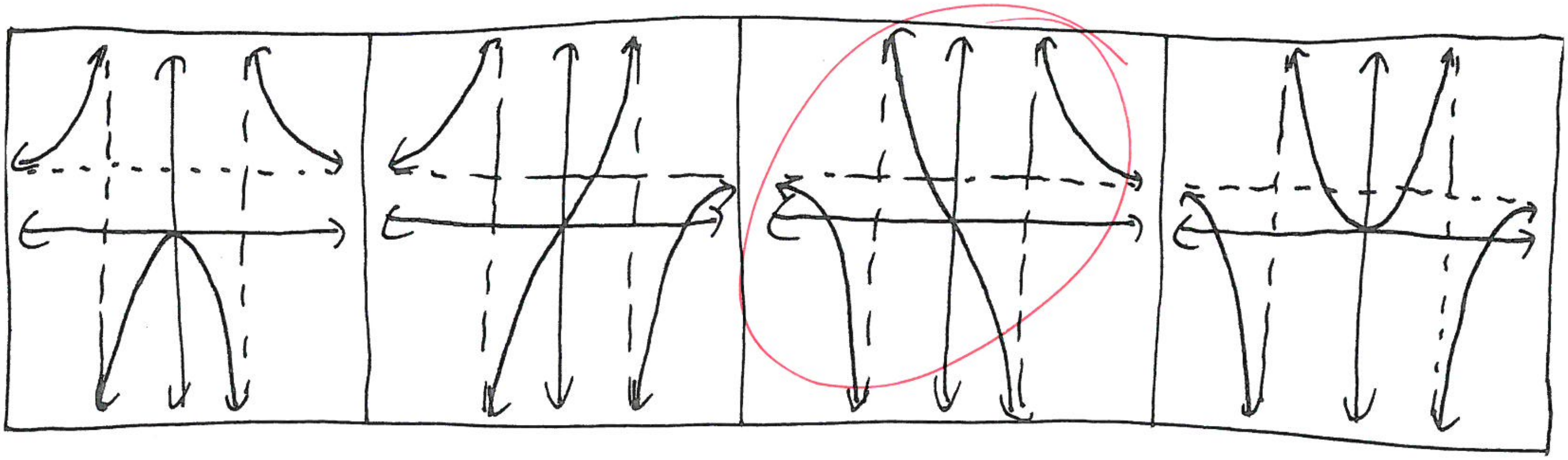
VA: need lim of $\pm\infty$

happens when denom. $= 0$

$$x^2-9=0 \longrightarrow x^2=9 \longrightarrow \boxed{x=\pm 3}$$

$$\lim_{x\to 3^+} \frac{2x^2+7x}{x^2-9} = \frac{39}{0^+} = \infty \qquad \lim_{x\to -3^+} \frac{2x^2+7x}{x^2-9} = \frac{-3}{0^-} = +\infty$$

$$\lim_{x\to 3^-} \frac{2x^2+7x}{x^2-9} = \frac{39}{0^-} = -\infty \qquad \lim_{x\to -3^-} \frac{2x^2+7x}{x^2-9} = \frac{-3}{0^+} = -\infty$$



4. (8 pts. each) Find the derivatives of the following functions by using the differentiation rules we learned in class. YOU DO NOT NEED TO SIMPLIFY YOUR ANSWERS!!!

(i) $h(x) = \sin(\cos x + \tan x)$

$$h'(x) = \cos(\cos x + \tan x)\cdot(\cos x + \tan x)'$$
$$= \boxed{\cos(\cos x + \tan x)\cdot(-\sin x + \sec^2 x)}$$

(ii) $h(x) = \dfrac{e^{2x} + e^{3x}}{e^{4x} + e^{5x}}$

$$h'(x) = \frac{(e^{2x}+e^{3x})'(e^{4x}+e^{5x}) - (e^{2x}+e^{3x})(e^{4x}+e^{5x})'}{(e^{4x}+e^{5x})^2}$$
$$= \boxed{\frac{(2e^{2x}+3e^{3x})(e^{4x}+e^{5x}) - (e^{2x}+e^{3x})(4e^{4x}+5e^{5x})}{(e^{4x}+e^{5x})^2}}$$

(iii) $h(x) = \underbrace{(x + x^{-1})(x^2 + x^{-2})}_{F}\underbrace{(x^3 + x^{-3})}_{g}$

$$= [(x+x^{-1})(x^2+x^{-2})]'\cdot(x^3+x^{-3}) + (x+x^{-1})(x^2+x^{-2})\cdot(x^3+x^{-3})'$$
$$= [(x+x^{-1})'(x^2+x^{-2}) + (x+x^{-1})(x^2+x^{-2})']\cdot(x^3+x^{-3})$$
$$+ (x+x^{-1})(x^2+x^{-2})(3x^2-3x^{-4})$$
$$= [(1-x^{-2})(x^2+x^{-2}) + (x+x^{-1})(2x-2x^{-3})](x^3+x^{-3})$$
$$+ (x+x^{-1})(x^2+x^{-2})(3x^2-3x^{-4})$$

5. (15 pts.) Find all x-values in $[0, 2\pi]$ where the tangent line to $y = (\sin x)^2$ is horizontal.

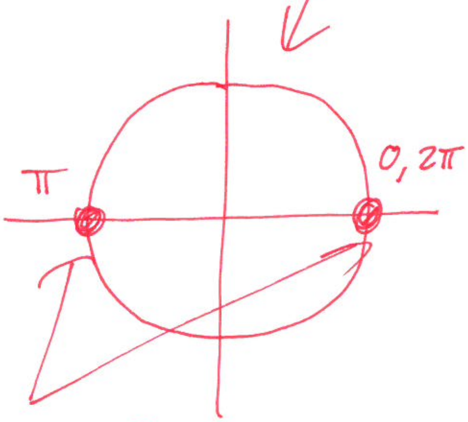
means set $y' = 0$

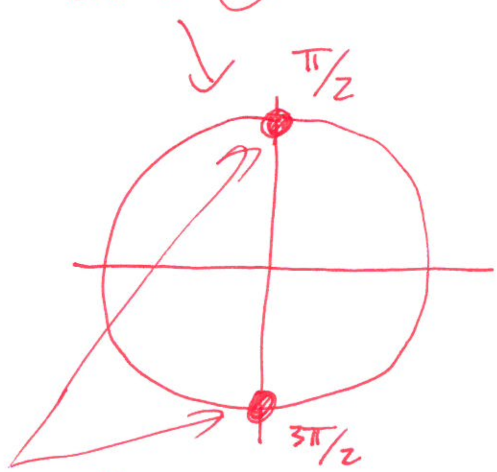
$$y' = 2(\sin x)(\sin x)'$$
$$= 2\sin x \cos x$$

So, $y' = 0$

$$2\sin x \cos x = 0$$

$\sin x = 0$ $\qquad$ $\cos x = 0$





$\sin x = 0$
(y-coord. is sine)

$\cos x = 0$
(x-coord. is cosine)

So, answers are $\boxed{x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi,}$
(there are no more answers since we said x had to be in $[0, 2\pi]$)

6. (12 pts; 4 pts. each) You are running a lemonade stand, and the function $P(x)$ represents your profit (in dollars) in a given day if you produce x gallons of lemonade to sell that day. Suppose that $P(30) = 40$ and $P'(30) = 2$.

(i) Write, in your own words, the meaning of the equation "$P'(30) = 2$." This does not need to be a paragraph, but be as clear as possible.

$P'(30)$ represents rate of change of P when $x = 30$, in units of $\frac{P\text{-unit}}{x\text{-unit}} = \$/\text{gallon}$

So, $P'(30) = 2$ means profit changing at \$2/gallon when 30 gallons are made.

(ii) Based on the information you have, ESTIMATE your daily profit if you produced 33 gallons of lemonade to sell in a day. Explain your answer!

At 30 gallons, profit was $P(30) = \$40$. At this production, profit increasing at \$2/gallon, So, 3 additional gallons should yield about $3 \cdot \$2 = \$6$ in additional profit, or $\boxed{\$46}$ total.

(iii) If you had to ESTIMATE the number of gallons of lemonade you'd need to produce to have a profit of \$50 in a given day, what would your guess be? Explain your answer!

You need \$10 additional in profit from the \$40 you make at $x = 30$. At \$2/gal, this should take $\$10/\$2 = 5$ more gallons, or $\boxed{35}$ total.