MATH 1951 Midterm Exam 2

Name: Solutions

Class time: (circle the one for your class) 11:00 - 12:00 12:00 - 1:00

**Instructions:** This test should have 6 pages and 5 problems, and is out of 100 points. Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this exam. (Approved: non-graphing, non-programmable, doesn't take derivatives)

**1.** (10 pts.) Find the derivative of $f(x) = x^x$ by using logarithmic differentiation.

$$\ln f = \ln(x^x)$$
$$\ln f = x \ln x$$

$$(\ln f)' = (x)' \ln x + x(\ln x)'$$
$$= (\ln x) + x\left(\frac{1}{x}\right)$$
$$= (\ln x) + 1$$

$$f' = f \cdot (\ln f)'$$
$$\boxed{f' = x^x \cdot ((\ln x) + 1)}$$

**2.** (10 pts.) Use linear approximation to find an approximate value for $\sqrt{98}$, and put your answer in decimal form.

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$x = 98$$

$$a = 100$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$\sqrt{98} \approx \sqrt{100} + \frac{1}{2\sqrt{100}}(98-100)$$

$$= 10 + \frac{1}{20}(-2)$$

$$= 10 - \frac{1}{10}$$

$$= \boxed{9.9}$$

**3.** (15 pts.) A sample of carbon-12 decays from 30 grams to 5 grams in 6 hours. How long will it take to decay to just 1 gram? (If you simplify your answer, round it to one decimal place)

$$P(t) = Ce^{kt}$$

$$C = 30$$

$$P(6) = 5 \quad \text{so}$$

$$5 = 30e^{k(6)}$$

$$\frac{1}{6} = e^{6k}$$

$$\ln\left(\frac{1}{6}\right) = 6k$$

$$k = \frac{\ln \frac{1}{6}}{6}$$

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To solve $P(t) = 1$:

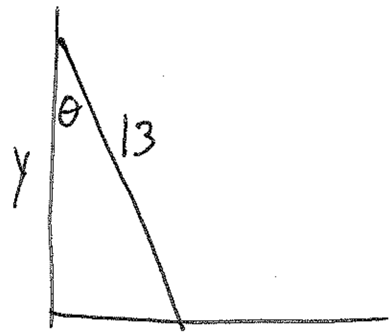
$$1 = 30e^{\left(\frac{\ln \frac{1}{6}}{6}\right)t}$$

$$\frac{1}{30} = e^{\left(\frac{\ln \frac{1}{6}}{6}\right)t}$$

$$\ln \frac{1}{30} = \left(\frac{\ln \frac{1}{6}}{6}\right)t$$

$$\boxed{t = \frac{\ln \frac{1}{30}}{\left(\frac{\ln \frac{1}{6}}{6}\right)} \approx 11.39 \text{ hours}}$$

4. (25 pts.) A ladder 13 feet long leans on a wall next to a corner. The top of the ladder slides down the wall at a rate of 3 feet per second. How fast is the angle between the wall and the top of the ladder changing when the top of the ladder is 5 feet from the ground?




$$\frac{dy}{dt} = -3$$

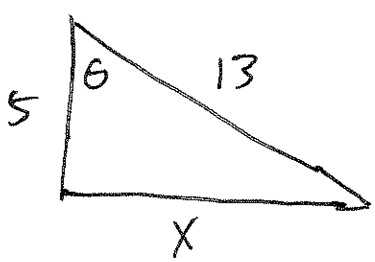
Find $\frac{d\theta}{dt}$ when $y=5$

$$\cos\theta = \frac{y}{13}$$

$$13\cos\theta = y$$

$$-13\sin\theta \frac{d\theta}{dt} = \frac{dy}{dt}$$

Need $\sin\theta$ when $y=5$:




$$5^2+x^2=13^2$$
$$25+x^2=169$$
$$x^2=144$$
$$x=\pm 12$$

Only $x=12$ makes sense

So $\sin\theta = \frac{12}{13}$. Plug in:

$$-13\left(\frac{12}{13}\right)\frac{d\theta}{dt} = -3$$

$$-12\frac{d\theta}{dt} = -3 \longrightarrow \frac{d\theta}{dt} = \frac{-3}{-12} = \boxed{\frac{1}{4} \text{ rad/sec}}$$

5. (40 pts.) Sketch a graph of $\frac{x^2-9}{x^2+3}$, making sure to include any $x$- and $y$-intercepts, horizontal and vertical asymptotes, local minimums and maximums, inflection points, and intervals where $f$ is increasing, decreasing, concave up, and concave down. (SHOW YOUR WORK for these items!) You may use the back of this page if you need more room.

X-intercept: $\frac{x^2-9}{x^2+3}=0 \rightarrow x^2-9=0$

$x^2=9$

$x=\pm 3$

$(-3, 0)$

$(3, 0)$

y-intercept: plug in $x=0$

$\frac{0^2-9}{0^2+3}=\frac{-9}{3}=-3 \quad (0, -3)$

Horiz. asymptote: $\lim_{x\to\pm\infty}\frac{x^2-9}{x^2+3}=\lim_{x\to\pm\infty}\frac{\frac{x^2}{x^2}-\frac{9}{x^2}\to 0}{\frac{x^2}{x^2}+\frac{3}{x^2}\to 0}=\frac{1}{1}=1$

Horiz. asymptote at $y=1$

Vertical asymptote: $\frac{x^2-9}{x^2+3}$ DNE $\rightarrow x^2+3=0$

$x^2=-3$

~~$x=\pm\sqrt{-3}$~~

No vertical asymptote

$$f'(x) = \frac{(x^2-9)'(x^2+3) - (x^2-9)(x^2+3)'}{(x^2+3)^2}$$

$$= \frac{(2x)(x^2+3) - (x^2-9)(2x)}{(x^2+3)^2} = \frac{2x^3+6x-2x^3+18x}{(x^2+3)^2}$$

$$= \frac{24x}{(x^2+3)^2}$$

$f' = 0$: $24x = 0$

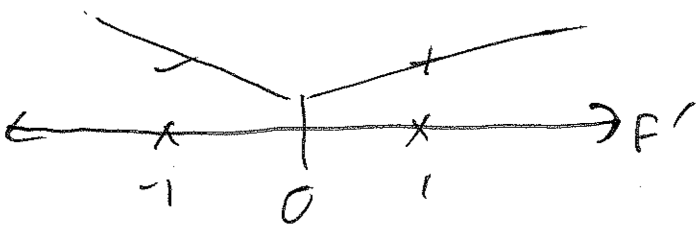
$x = 0$

$f'$ DNE: $(x^2+3)^2 = 0$

$x^2+3 = 0$

$x^2 = -3$

~~$x = \pm\sqrt{-3}$~~




Local min at $x=0$, $y=-3$ $(0, -3)$

$$f''(x) = \frac{(24x)'(x^2+3)^2 - (24x)((x^2+3)^2)'}{((x^2+3)^2)^2}$$

$$= \frac{24(x^2+3)^2 - 24x \cdot 2(x^2+3)(2x)}{(x^2+3)^4}$$

$$= \frac{24(x^2+3)[x^2+3-4x^2]}{(x^2+3)^4}$$

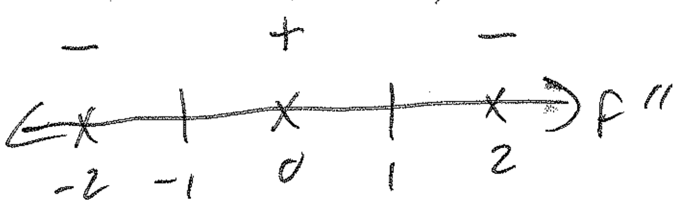
$$= \frac{24(3-3x^2)}{(x^2+3)^3}$$

$f''=0$: $3-3x^2=0$

$$3=3x^2$$
$$x^2=1$$
$$x=\pm 1$$

$f''$ DNE: $(x^2+3)^3=0$

$$x^2+3=0$$
$$x^2=-3$$
$$\sout{x=\pm\sqrt{-3}}$$




Inflection pts at

$x=-1$, $x=1$

At $x=-1$, $y=\frac{(-1)^2-9}{(-1)^2+3}$

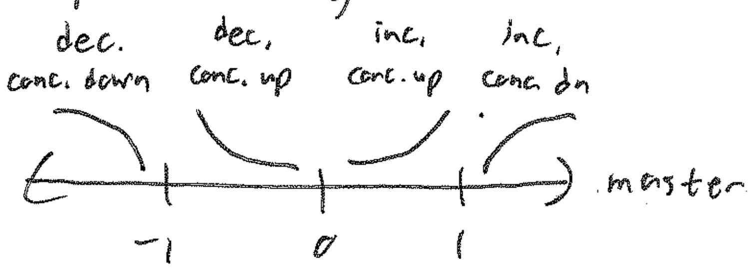
$$=\frac{-8}{4}=-2$$

At $x=1$, $y=\frac{1^2-9}{1^2+3}$

$$=\frac{-8}{4}=-2$$

$(-1,-2)$, $(1,-2)$ inflection points

Master sign chart:



Graph: