Practice ✓

MATH 1951 Final Exam

Name: Solutions

Class time: (circle the one for your class) ~~11:00 - 12:00~~ 1:00 - 2:00

Instructions: This test should have 12 pages and 12 problems, and is out of 100 points. Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this exam. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. (6 pts.) Compute the derivative of $f(x) = \frac{5}{3-2x}$ at $x = -1$ by using the LIMIT DEFINITION of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{5}{3-2(x+h)} - \frac{5}{3-2x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{5(3-2x) - 5(3-2(x+h))}{(3-2(x+h))(3-2x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{15} - \cancel{10x} - \cancel{15} + \cancel{10x} + 10h}{(3-2(x+h))(3-2x)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{10\cancel{h}}{(3-2(x+h))(3-2x)} \cdot \frac{1}{\cancel{h}}$$

$$= \frac{10}{(3-2(x+0))(3-2x)} = \boxed{\frac{10}{(3-2x)^2}}$$

2. (6 pts.) Find the absolute/global maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 12x + 1$ on the domain $[-2, 0]$.

$f' = 6x^2 - 6x + 12$

Critical #s:

~~f' DNE~~

$f' = 0: \quad 6x^2 - 6x + 12 = 0$

$6(x^2 - x + 2) = 0$

$6(x+1)(x-2) = 0$

$x = -1, \cancel{2} \leftarrow$ not in domain

Check crit. #s and endpoints

$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1$

$= -16 - 12 + 24 + 1 = -3$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1 = 8$

$f(0) = 2 \cdot 0^3 - 3 \cdot 0^2 - 12 \cdot 0 + 1 = 1$

$$\boxed{\begin{array}{l} \text{Global min: } -3, \text{ at } x = -2 \\ \text{Global max: } 8, \text{ at } x = -1 \end{array}}$$

3. (5 pts.) Use logarithmic differentiation to find the derivative of
$f(x) = \dfrac{(3x+1)^{10}e^{\sqrt{x}}}{(\sin x)^5}$.

$$\ln f = \ln\left[\frac{(3x+1)^{10}e^{\sqrt{x}}}{(\sin x)^5}\right]$$

$$= \ln\left((3x+1)^{10}\right) + \cancel{\ln}\left(\cancel{e}^{\sqrt{x}}\right) - \ln\left((\sin x)^5\right)$$

$$= 10\ln(3x+1) + \sqrt{x} - 5\ln(\sin x)$$

$$(\ln f)' = 10 \cdot \frac{1}{3x+1} \cdot 3 + \frac{1}{2}x^{-1/2} - 5 \cdot \frac{1}{\sin x} \cdot \cos x$$

$$= \frac{30}{3x+1} + \frac{1}{2\sqrt{x}} - \frac{5\cos x}{\sin x}$$

$$f' = f \cdot ((\ln f)') = \left[\frac{(3x+1)^{10} \cdot e^{\sqrt{x}}}{(\sin x)^5}\right] \cdot \left[\frac{30}{3x+1} + \frac{1}{2\sqrt{x}} - \frac{5\cos x}{\sin x}\right]$$

4. (15 pts.; 5 pts. each) Find the derivatives of the following functions. You do NOT need to simplify your answers.

(a) $h(x) = \underbrace{\sin x \sin(x^2)}_{f}\,\underbrace{\sin(x^3)}_{g}$

$$h' = (\sin x \cdot \sin(x^2))' \cdot \sin(x^3) + (\sin x \cdot \sin(x^2)) \cdot (\sin(x^3))'$$

$$= \left[(\sin x)' \sin(x^2) + \sin x \cdot (\sin(x^2))'\right] \sin(x^3) + \sin x \cdot \sin(x^2) \cdot \cos(x^3) \cdot (x^3)'$$

$$= \boxed{\left[\cos x \cdot \sin(x^2) + \sin x \cdot \cos(x^2) \cdot 2x\right] \sin(x^3) + \sin x \cdot \sin(x^2) \cdot \cos(x^3) \cdot 3x^2}$$

(b) $g(x) = \dfrac{3^x + 4^{-x}}{4^x - 3^{-x}}$

$$g' = \frac{(3^x + 4^{-x})'(4^x - 3^{-x}) - (3^x + 4^{-x})(4^x - 3^{-x})'}{(4^x - 3^{-x})^2}$$

$$= \boxed{\frac{(3^x \ln 3 + 4^{-x}(\ln 4) \cdot (-1))(4^x - 3^{-x}) - (3^x + 4^{-x})(4^x \ln 4 - 3^{-x}(\ln 3)(-1))}{(4^x - 3^{-x})^2}}$$

(c) $f(x) = \sqrt{\arctan x} + \arctan \sqrt{x}$

$$f = (\arctan x)^{1/2} + \arctan(x^{1/2})$$

$$f' = \frac{1}{2}(\arctan x)^{-1/2} \cdot (\arctan x)' + \frac{1}{1 + (x^{1/2})^2} \cdot (x^{1/2})'$$

$$= \boxed{\frac{1}{2\sqrt{\arctan x}} \cdot \frac{1}{1+x^2} + \frac{1}{1+x} \cdot \frac{1}{2} x^{-1/2}}$$

5. (8 pts.) Find the intervals where the function $f(x) = 4x - \tan x$ is increasing and decreasing within the domain $[0, 2\pi]$.

$F' = 4 - \sec^2 x$

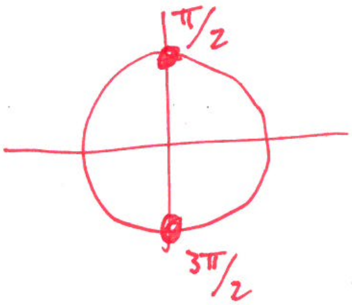
Sign chart

F' DNE:

$\sec x$ DNE

if $\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$



$F' = 0$:

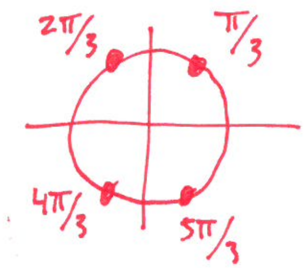
$4 - \sec^2 x = 0$

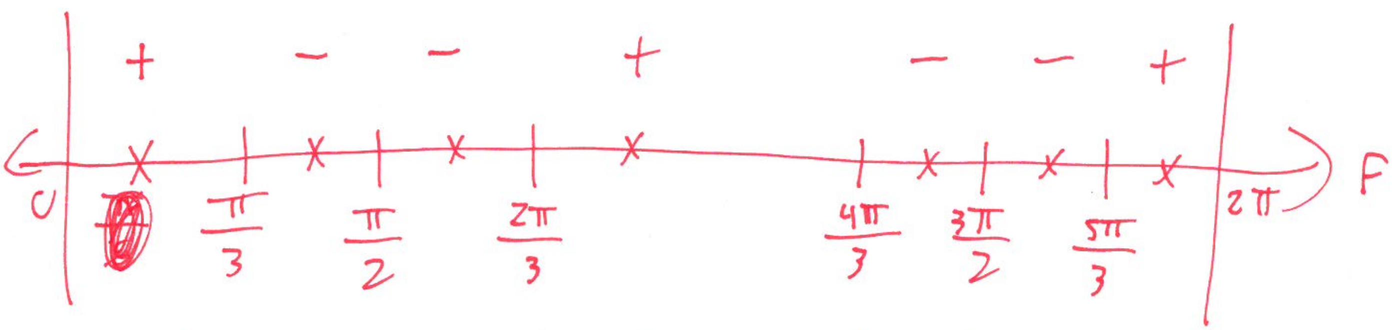
$4 = \sec^2 x$

$\pm 2 = \sec x$

$\cos x = \pm \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$





choose any test values, with calculator if you wish

$\boxed{\begin{array}{l} \text{inc. on } [0, \frac{\pi}{3}), (\frac{2\pi}{3}, \frac{4\pi}{3}), (\frac{5\pi}{3}, 2\pi] \\ \text{dec. on } (\frac{\pi}{3}, \frac{2\pi}{3}), (\frac{4\pi}{3}, \frac{5\pi}{3}) \end{array}}$

6. (8 pts.) Find all horizontal and vertical asymptotes of $f(x) = \dfrac{x^{\frac{4}{3}} + 2}{x^{\frac{1}{3}} - x^{\frac{4}{3}}}$.

HA:

$$\lim_{x \to \pm\infty} \frac{x^{4/3} + 2}{x^{1/3} - x^{4/3}} = \lim_{x \to \pm\infty} \frac{\frac{x^{4/3}}{x^{4/3}} + \frac{2}{x^{4/3}} \to 0}{\frac{x^{1/3}}{x^{4/3}} - \frac{x^{4/3}}{x^{4/3}}}$$

$$\to 0$$

$$= \frac{1}{-1} = -1 \quad \boxed{\text{HA at } y = -1}$$

VA:

Where lim of $\dfrac{x^{4/3} + 2}{x^{1/3} - x^{4/3}}$ is $\pm\infty$:

happens when denominator $= 0$

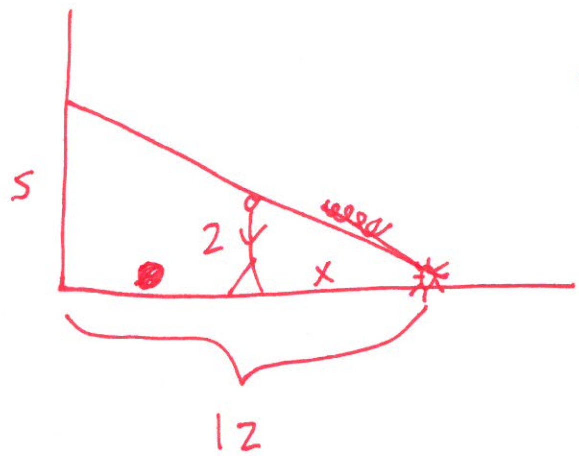
$$x^{1/3} - x^{4/3} = 0$$

$$x^{1/3}(1 - x) = 0$$

$x^{1/3} = 0$ | $1 - x = 0$
$\sqrt[3]{x} = 0$ | $x = 1$
$x = 0$

$$\boxed{\text{VA at } x = 0,\ x = 1}$$

7. (15 pts.) A spotlight on the ground shines on a wall 12 m away. A man 2 m tall walks from the spotlight towards the wall at a rate of 3 m/s. How fast is the length of his shadow on the building changing when he is 2 feet from the spotlight?



$\frac{dx}{dt} = +3$

Find $\frac{ds}{dt}$ when $x = 2$

Similar triangles: $\frac{2}{x} = \frac{s}{12}$

$2x^{-1} = \frac{1}{12} s$

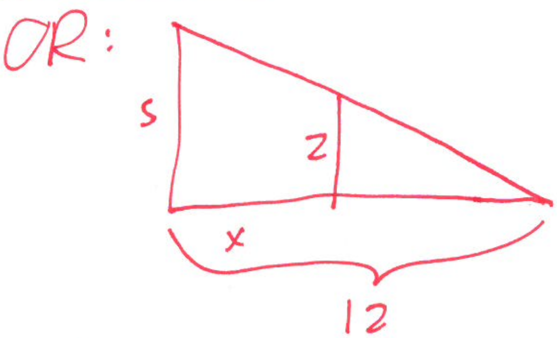
$-2x^{-2} \frac{dx}{dt} = \frac{1}{12} \frac{ds}{dt}$

Plug in: $-2 \cdot \frac{1}{2^2} \cdot 3 = \frac{1}{12} \frac{ds}{dt}$

$-\frac{3}{2} = \frac{1}{12} \frac{ds}{dt}$

$\boxed{\frac{ds}{dt} = -18 \text{ m/s}}$

OR:



$\frac{dx}{dt} = -3$

Find $\frac{ds}{dt}$ when $x = 10$

$\frac{2}{12-x} = \frac{s}{12}$

$2(12-x)^{-1} = \frac{1}{12} s$

$-2(12-x)^{-2}\left(-\frac{dx}{dt}\right) = \frac{1}{12} \frac{ds}{dt}$

Plug in:

$-2(12-10)^{-2}(-(-3)) = \frac{1}{12} \frac{ds}{dt}$

$-2 \cdot 2^{-2} \cdot 3 = \frac{1}{12} \frac{ds}{dt}$

Still, $\frac{ds}{dt} = -18$ m/s

8. (6 pts.) Use linear approximation to approximate $\sqrt[3]{8.1}$.

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\quad f'(x) = \frac{1}{3}x^{-2/3}$

$x = 8.1$ $\quad = \frac{1}{3\sqrt[3]{x^2}}$

$a = 8$

$f(a) = \sqrt[3]{8} = 2$ $\quad f'(a) = \frac{1}{3\sqrt[3]{64}} = \frac{1}{12}$

$f(x) \approx f(a) + (x-a) \cdot f'(a)$

$\sqrt[3]{8.1} \approx 2 + 0.1 \cdot \frac{1}{12} = 2 + \frac{1}{120} = \boxed{\frac{241}{120}}$

$= 2.0083333\ldots$

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~~8.~~ (15 pts.) A cylinder has surface area 24π square feet. What is the maximum possible volume of such a cylinder? (Hint: the surface area of a cylinder is $2\pi r^2 + 2\pi rh$.)

~~$A = 2\pi r^2 + 2\pi rh$~~

~~Constraint: Volume =~~

$V = \pi r^2 h$

Constraint: Surface Area = 24

$2\pi r^2 + 2\pi rh = 24$

Solve for h:

$2\pi rh = 24 - 2\pi r^2$

$h = \frac{24 - 2\pi r^2}{2\pi r}$

$h = \frac{12}{\pi r} - r$

Plug in to V:

$V = \pi r^2 h = \pi r^2 \left(\frac{12}{\pi r} - r\right)$

$= 12r - \pi r^3$

Domain:

r in $\left(0, \sqrt{\frac{12}{\pi}}\right)$

b/c at $r = \sqrt{\frac{12}{\pi}}$, top & bot of cylinder already use all 24π of allowed area

$V' = 12 - 3\pi r^2$

~~V' DNE~~ $V' = 0$: $12 - 3\pi r^2 = 0$

$12 = 3\pi r^2$

$r^2 = \frac{4}{\pi}$

$r = \cancel{\pm}\sqrt{\frac{4}{\pi}}$

Find V

$V = 12r - \pi r^3$

$= 12\sqrt{\frac{4}{\pi}} - \pi\left(\sqrt{\frac{4}{\pi}}\right)^3$

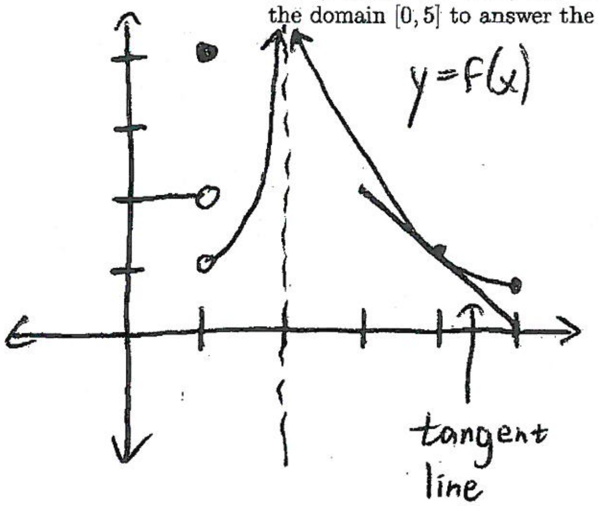
$= \frac{24}{\sqrt{\pi}} - \pi \cdot \frac{8}{\pi \cdot \sqrt{\pi}}$

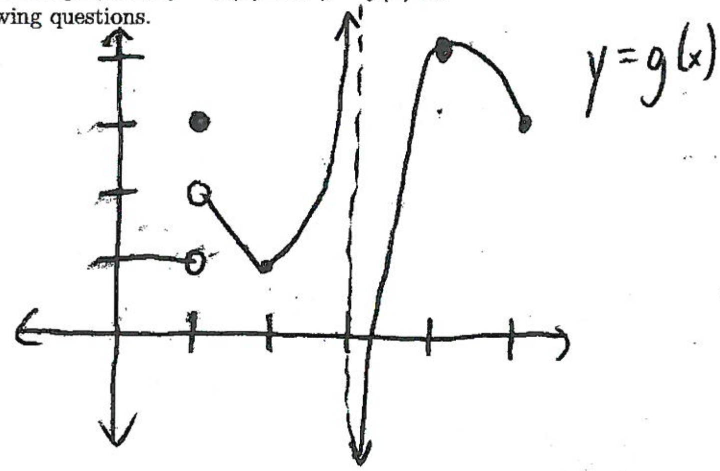
$= \boxed{\frac{16}{\sqrt{\pi}} \text{ ft}^3}$

$V'' = -6\pi r$

$V''\left(\sqrt{\frac{4}{\pi}}\right) < 0$, so $r = \sqrt{\frac{4}{\pi}}$ is local max, also global since only one

10. (8 pts.; 1 pt. each) Use the provided graphs of $y = f(x)$ and $y = g(x)$ on the domain $[0, 5]$ to answer the following questions.





(a) For what values of x is $f(x)$ discontinuous?

$\boxed{x = 1, 2}$

(b) For what values of x is $g(x)$ nondifferentiable?

$\boxed{x = 1, 2, 3}$

(c) Find $\lim_{x\to 1^-}(f(x) + g(x))$.

$= \lim_{x\to 1^-} f(x) + \lim_{x\to 1^-} g(x) = 2 + 1 = \boxed{3}$

(d) Find $\lim_{x\to 1^+}(f(x) + g(x))$.

$\lim_{x\to 1^+} f(x) + \lim_{x\to 1^+} g(x) = 1 + 2 = \boxed{3}$

(e) Is $f(x) + g(x)$ continuous at $x = 1$?

$\boxed{\text{No:}}$ $f(1) + g(1) = 4 + 3 = 7$, not equal to $\lim_{x\to 1} f(x) + g(x)$

(f) Find the derivative of $f(x) - g(x)$ at $x = 4$.

$(f - g)' = f' - g'$

$f'(4) - g'(4) = -1 - 0 = \boxed{-1}$

(g) Find the derivative of $f(x)g(x)$ at $x = 4$.

$(fg)' = f'g + fg'$

$f'(4) \cdot g(4) + f(4) \cdot g'(4)$

$= -1 \cdot 4 + 1 \cdot 0$

$= \boxed{-4}$

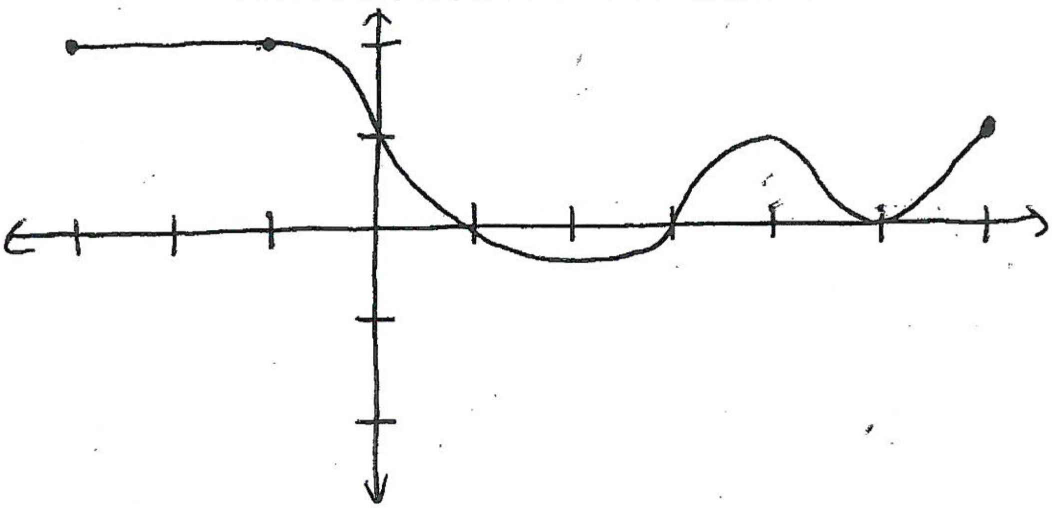
(h) Find the derivative of $f(g(x))$ at $x = 4$.

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$f'(g(4)) \cdot g'(4)$

$= f'(4) \cdot 0 = -1 \cdot 0 = \boxed{0}$

11. (6 pts.; 1 pt. each) Use the provided graph of $y = f'(x)$ (NOT $y = f(x)$!!!) on the domain $[-3, 6]$ to answer the following questions.



(a) On what intervals is $f(x)$ decreasing?

f dec when $f' < 0$: $(1, 3)$

(b) At what x-values does $f(x)$ have local extrema?

f has extrema when f' changes sign: $x = 1$, $x = 3$

(c) On what intervals is $f(x)$ concave up?

f conc. up when $(f')' > 0$: $(2, 4)$ and $(5, 6)$

(d) At what x-values does $f(x)$ have inflection points?

f has inf. pt. when $(f')'$ changes sign: $x = 2$, $x = 4$, $x = 5$

(e) What is $f(-1) - f(-3)$?

f' constantly 2 on $(-3, -1)$, so f has slope 2 on $(-3, -1)$, so

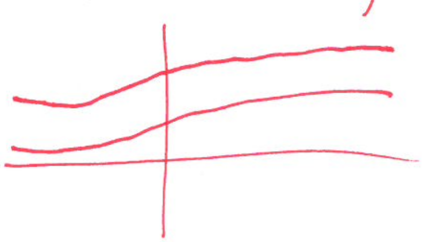
$$\frac{f(-1) - f(-3)}{(-1) - (-3)} = 2$$

$$\downarrow$$

$$f(-1) - f(-3) = 4$$

(f) In your own words, why isn't it possible to draw an accurate graph of $y = f(x)$, even with exact information about all values of $f'(x)$?

Because f' only gives info about the slope of f, not its y-coordinates. The following two graphs



would have the same f', so f' can't help you distinguish btwn. them

12. (6 pts.) Show that the function $f(x) = x^5 + x^2 + x + 1$ has some tangent line with slope 3. You do NOT need to solve for the x-value where this tangent line occurs. (Hint: you can do this with either the Intermediate Value Theorem OR the Mean Value Theorem)

Using IVT:

$f' = 5x^4 + 2x + 1$

f' continuous on $(-\infty, \infty)$

$f'(0) = 1 < 3$

$f'(1) = 8 > 3$

So, by IVT, there's a c btwn. $x=0$, $x=1$ where $f'(c) = 3$; this means that f has tangent line w/slope 3 there