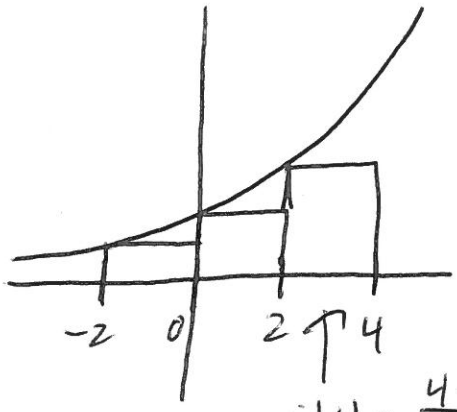
MATH 1952 Midterm Exam 1

Name: Solutions

Instructions: This test should have 7 problems on 6 pages, and is out of 100 points. Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this exam. (Approved: non-graphing, non-programmable, doesn't take derivatives) Answers do not need to be fully simplified unless otherwise specified in the problem.

1. Find approximations to the definite integral $\int_{-2}^{4} 2^x \, dx$ in the following ways.

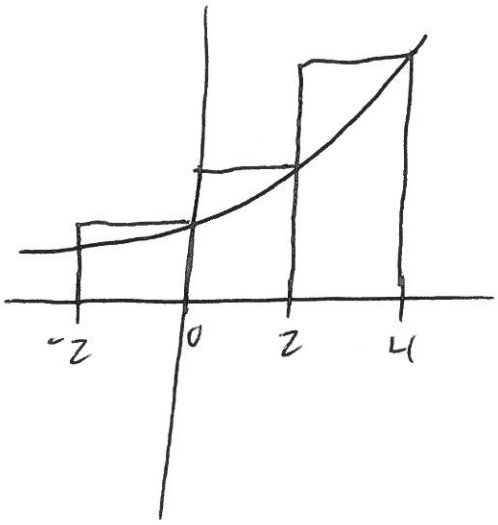
(a) (8 pts.) Riemann sum with 3 rectangles using left endpoints



$$\text{width} = \frac{4-(-2)}{3} = \frac{6}{3} = 2$$

$$2(f(-2)+f(0)+f(2))$$
$$= 2(2^{-2}+2^{0}+2^{2})$$
$$= 2\left(\frac{1}{4}+1+4\right) = \boxed{\frac{21}{2}}$$

(b) (8 pts.) Riemann sum with 3 rectangles using right endpoints



$$2(f(0)+f(2)+f(4))$$
$$= 2(2^{0}+2^{2}+2^{4})$$
$$= 2(1+4+16) = \boxed{42}$$

2. Suppose that for two mystery functions f and g, you know that $\int_0^1 f(x)\,dx = 5$, $\int_0^2 f(x) = 4$, and $\int_0^2 g(x)\,dx = 7$. Find the following integrals.

(a) (4 pts.) $\int_0^1 (2 + f(x))\,dx$

$$= \int_0^1 2\,dx + \int_0^1 f(x)\,dx$$

$$= 2x\Big|_0^1 + 5 = 2 - 0 + 5 = \boxed{7}$$

(b) (4 pts.) $\int_1^2 f(x)\,dx$

$$= \int_0^2 f(x)\,dx - \int_0^1 f(x)\,dx$$

$$= 4 - 5 = \boxed{-1}$$

(c) (4 pts.) $\int_0^2 2f(x) + 3g(x)\,dx$

$$= 2\int_0^2 f(x)\,dx + 3\int_0^2 g(x)\,dx$$

$$= 2 \cdot 4 + 3 \cdot 7 = \boxed{29}$$

3. Find the following integrals, which should not require u-substitution. You DO NOT need to simplify your answers.

(a) (10 pts.) $\int \frac{\sqrt{x} - x}{x^2}\, dx$

$$= \int \frac{x^{1/2} - x^1}{x^2}\, dx = \int \left(x^{1/2} - x^1\right) x^{-2}\, dx$$

$$= \int x^{-3/2} - x^{-1}\, dx$$

$$= \frac{x^{-1/2}}{-1/2} - \ln|x| + C$$

$$= \boxed{\frac{-2}{\sqrt{x}} - \ln|x| + C}$$

(b) (10 pts.) $\int_0^1 e^x + x^e + e^e\, dx$

$$= e^x + \frac{x^{e+1}}{e+1} + e^e x \Big|_0^1$$

$$= e^1 + \frac{1^{e+1}}{e+1} + e^e \cdot 1 - \left(e^0 + \frac{0^{e+1}}{e+1} + e^e \cdot 0\right)$$

$$= \boxed{e + \frac{1}{e+1} + e^e - 1}$$

4. (5 pts.) Consider the following two integrals: $\int_{-1}^{0} \frac{1}{(3x+1)^2}\,dx$ and $\int_{0}^{1} \frac{1}{(3x+1)^2}\,dx$. One of them can NOT be computed via part 2 of the Fundamental Theorem of Calculus. Tell me which one can be computed, which one cannot, and why. You DO NOT need to actually find the value of any integrals for this problem.

$\frac{1}{(3x+1)^2}$ fails to exist if $3x+1=0$, i.e. if $x = -1/3$. So, you can't use FTC on $\int_{-1}^{0} \frac{1}{(3x+1)^2}\,dx$ since the bad value $x = -1/3$ is between $x=-1$ and $x=0$.

$\int_{0}^{1} \frac{1}{(3x+1)^2}\,dx$, however, is fine, since $\frac{1}{(3x+1)^2}$ is continuous on $[0, 1]$.

5. (5 pts.) Find the DERIVATIVE of the function $g(x) = \int_{x}^{4} e^{t^2}\,dt$ by using part 1 of the Fundamental Theorem of Calculus.

$$\left(\int_{x}^{4} e^{t^2}\,dt\right)' = \left(-\int_{4}^{x} e^{t^2}\,dt\right)'$$

$$= -\left(\int_{4}^{x} e^{t^2}\,dt\right)' = \boxed{-e^{x^2}}$$

6. Find the following integrals by using u-substitution. Please DO simplify these answers.

(a) (12 pts.) $\int_0^{\frac{\pi}{3}} (\sin x)(\cos x)^4 \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$dx = \frac{-1}{\sin x} du$

$= \int_1^{\frac{1}{2}} (\cancel{\sin x}) u^4 \cdot \frac{-1}{\cancel{\sin x}} du$

$= \int_1^{\frac{1}{2}} -u^4 \, du$

$= \frac{-u^5}{5} \Big|_1^{1/2} = \frac{-(1/2)^5}{5} - \frac{(-1)^5}{5} = \frac{\left(\frac{-1}{32}\right)}{5} - \left(\frac{-1}{5}\right)$

$= \frac{-1}{160} + \frac{1}{5} = \boxed{\frac{31}{160}}$

(b) (12 pts.) $\int \frac{\sec^2(\ln x)}{x} \, dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$dx = x \, du$

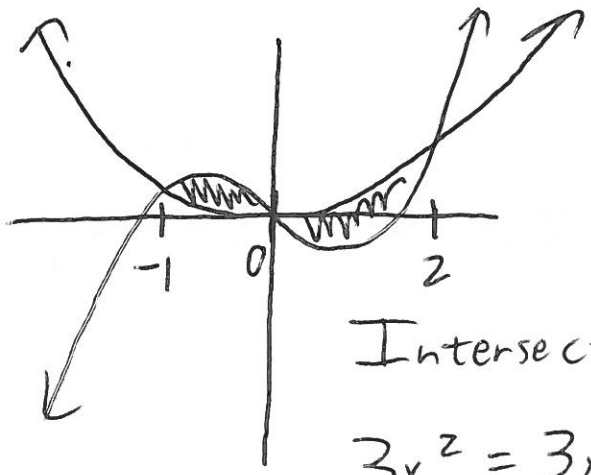
$= \int \frac{\sec^2 u}{\cancel{x}} \cdot \cancel{x} \, du$

$= \int \sec^2 u \, du$

$= \tan u + C$

$= \boxed{\tan(\ln x) + C}$

7. (18 pts.) Find the area trapped between the curves $f(x) = 3x^2$ and $g(x) = 3x^3 - 6x$ (you may refer to the picture below.)



Intersections:

$$3x^2 = 3x^3 - 6x$$
$$0 = 3x^3 - 3x^2 - 6x$$
$$0 = 3x(x^2 - x - 2)$$
$$0 = 3x(x-2)(x+1)$$
$$x = 0,\ 2,\ -1$$

Need to break into two integrals since upper & lower functions switch places

$$A = \int_{-1}^{0} \overbrace{3x^3 - 6x}^{\text{top}} - \overbrace{3x^2}^{\text{bottom}}\, dx + \int_{0}^{2} \overbrace{3x^2}^{\text{top}} - \overbrace{(3x^3 - 6x)}^{\text{bottom}}\, dx$$

$$= \frac{3}{4}x^4 - 3x^2 - x^3 \Big|_{-1}^{0} + x^3 - \left(\frac{3}{4}x^3 - 3x^2\right)\Big|_{0}^{2}$$

$$= 0 - 0 - 0 - \left(\frac{3}{4}(-1)^4 - 3(-1)^2 - (-1)^3\right) + 2^3 - \left(\frac{3}{4}\cdot 2^3 - 3\cdot 2^2\right) - (0 - (0 - 0))$$

$$= -\left(\frac{3}{4} - 3 + 1\right) - (6 - 12)$$

$$= -\left(\frac{-5}{4}\right) - (-6) = \frac{5}{4} + 6 = \boxed{\frac{29}{4}}$$