MATH 1952 Practice Final Exam

Name: Solutions

Instructions: This test should have 8 problems on 8 pages. Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this exam. (Approved: non-graphing, non-programmable, doesn't take derivatives) Answers do not need to be fully simplified unless otherwise specified in the problem.

1. At $t = 0$, a ball is thrown downwards at a speed of 32 feet per second from the lip of the Grand Canyon at a height of 1920 feet above the canyon floor.

(a) Using the fact that acceleration is given by $a(t) = -32$ feet per second squared, find a formula for the velocity $v(t)$ of the ball after t seconds. (Hint: what's $v(0)$?)

(b) Use your answer to (a) to give a formula for the height (or position) $h(t)$ of the ball after t seconds. (Hint: what's $h(0)$?) Use your answer to find out when the ball will hit the canyon floor.

(a) $v = \int a$

$v(t) = \int -32 \, dt \rightarrow v(t) = -32t + C$

But $v(0) = -32$ ft/s, so $C = -32$

$\boxed{v(t) = -32t - 32}$

(b) $h = \int v$

$h(t) = \int -32t - 32 \, dt \longrightarrow h(t) = -16t^2 - 32t + D$

But $h(0) = 1920$ ft, so $D = 1920$

$\boxed{h(t) = -16t^2 - 32t + 1920}$

When ball hits Floor, $h(t) = 0$

$-16t^2 - 32t + 1920 = 0$

$-16(t^2 + 2t - 120) = 0$

$-16(t+12)(t-10) = 0 \longrightarrow \cancel{t = -12}$ or $\boxed{t = 10\text{s}}$

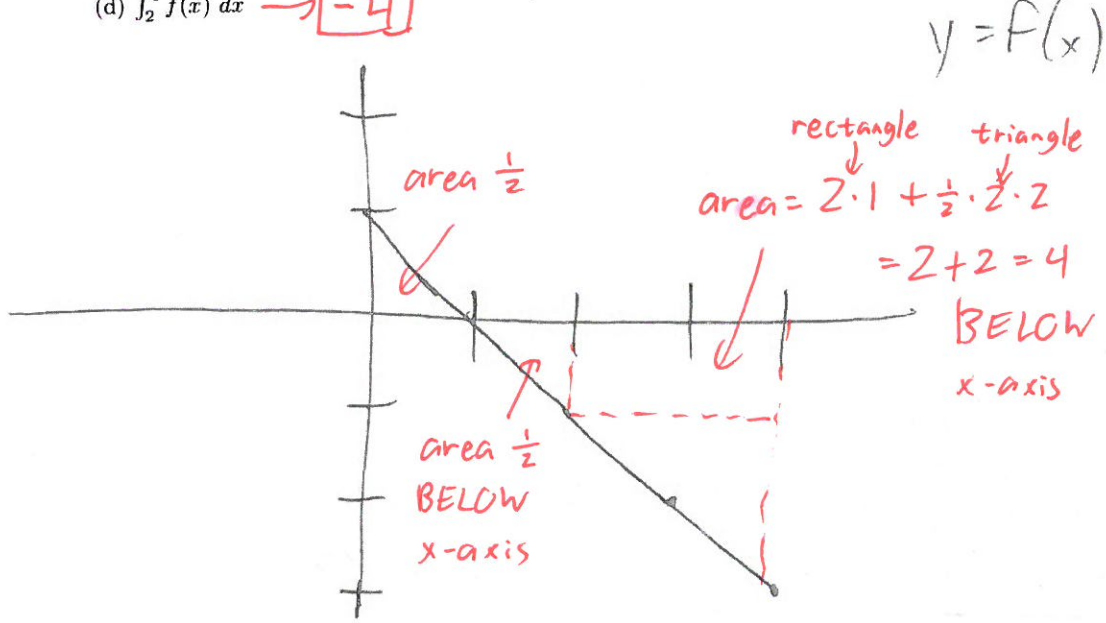
2. Using the pictures below, give the values of the following definite integrals:

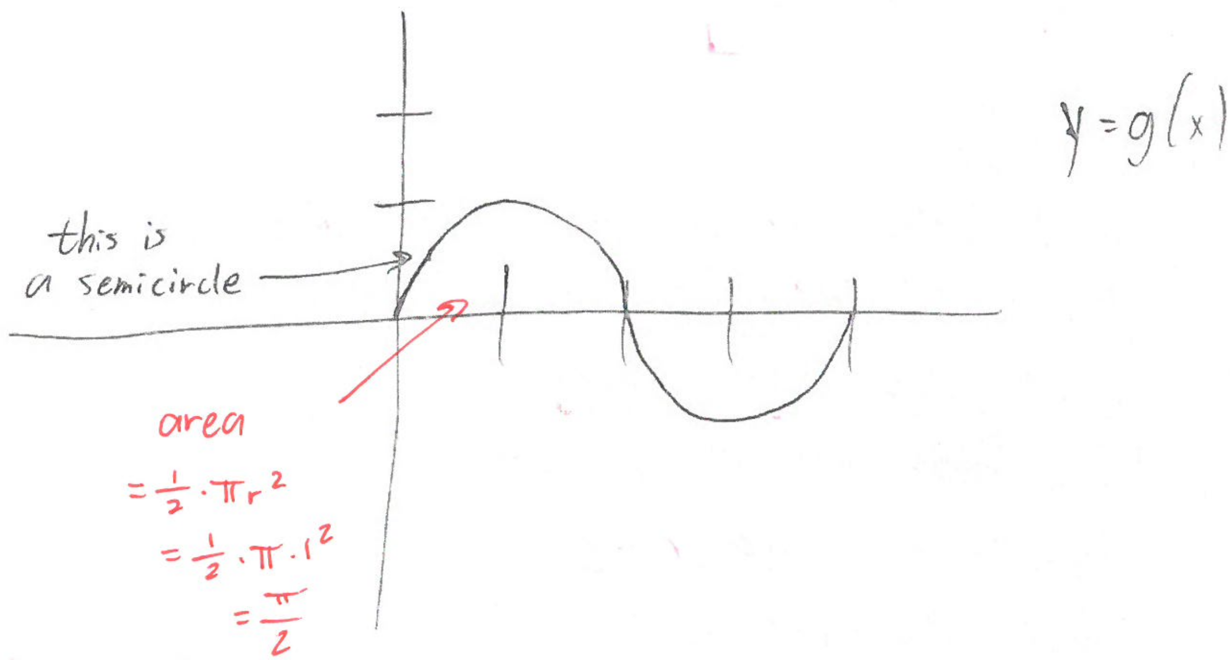
(a) $\int_0^2 f(x)\,dx$ → area $= +1/2 - 1/2 = \boxed{0}$

(b) $\int_0^2 g(x)\,dx$ → $\boxed{\pi/2}$

(c) $\int_0^2 2f(x) + 3g(x)\,dx$ → $2\int_0^2 f\,dx + 3\int_0^2 g\,dx = 2\cdot 0 + 3\cdot \pi/2 = \boxed{3\pi/2}$

(d) $\int_2^4 f(x)\,dx$ → $\boxed{-4}$





3. Find the area between the curves $y = x^2$ and $y = 8 - x^2$.

$x^2 = 8 - x^2$

$2x^2 = 8$

$x^2 = 4$ intersection pts.

$x = \pm 2$

Try test # btwn: at $x = 0$,

$x^2 = 0$ $8 - x^2 = 8$, so $8 - x^2 > x^2$

$$A = \int_{-2}^{2} (8 - x^2) - x^2 \, dx = \int_{-2}^{2} 8 - 2x^2 \, dx$$

$$= 8x - \frac{2}{3}x^3 \Big|_{-2}^{2} = 16 - \frac{2}{3} \cdot 8 - \left(-16 - \frac{2}{3}(-8)\right)$$

$$= 16 - \frac{16}{3} + 16 - \frac{16}{3}$$

$$= 32 - \frac{32}{3} = \boxed{\frac{64}{3}}$$

4. The curve $y = \ln x$ between $x = 1$ and $x = 3$ is rotated around the x-axis, yielding a solid. Find the volume of that solid.

Done in class

5. The curve $x = \frac{y^2}{8} - \ln y$ between $y = 1$ and $y = 3$ is rotated around the x-axis, yielding a solid. Find the surface area of that solid.

axis matches fcn, not variable.

So, $SA = 2\pi \int_1^3 y \cdot \sqrt{1 + (g'(y))^2}\, dy$

$g'(y) = \frac{y}{4} - \frac{1}{y}$

$g'(y)^2 = \frac{y^2}{16} - \frac{1}{2} + \frac{1}{y^2}$

$$SA = 2\pi \int_1^3 y \sqrt{1 + \frac{y^2}{16} - \frac{1}{2} + \frac{1}{y^2}}\, dy$$

$$= 2\pi \int_1^3 y \sqrt{\frac{y^2}{16} + \frac{1}{2} + \frac{1}{y^2}}\, dy$$

$$= 2\pi \int_1^3 y \sqrt{\left(\frac{y}{4} + \frac{1}{y}\right)^2}\, dy$$

$$= 2\pi \int_1^3 y \left(\frac{y}{4} + \frac{1}{y}\right) dy$$

$$= 2\pi \int_1^3 \frac{y^2}{4} + 1\, dy$$

$$= 2\pi \left(\frac{y^3}{12} + y \Big|_1^3\right) = 2\pi\left(\frac{3^3}{12} + 3 - \left(\frac{1^3}{12} + 1\right)\right)$$

$$= 2\pi\left(\frac{27}{12} + 3 - \frac{1}{12} - 1\right)$$

$$= 2\pi\left(\frac{26}{12} + 2\right) = 2\pi\left(\frac{25}{6}\right)$$

$$= \boxed{\frac{25\pi}{3}}$$

6. Find the average value of the function $f(x) = \frac{\sqrt{x^2-1}}{x^4}$ over the interval from $x = 1$ to $x = \sqrt{2}$.

$$\text{Avg} = \frac{1}{\sqrt{2}-1}\int_1^{\sqrt{2}} \frac{\sqrt{x^2-1}}{x^4}\,dx$$

$x = \sec t$

$\sqrt{x^2-1} = \tan t$

$dx = \sec t \tan t \, dt$

When $x=1$, $\sec t = 1$, $\frac{1}{\cos t} = 1$, $\cos t = 1$, $t = 0$

When $x=\sqrt{2}$, $\sec t = \sqrt{2}$, $\frac{1}{\cos t} = \sqrt{2}$, $\cos t = \frac{1}{\sqrt{2}}$, $t = \pi/4$

$$= \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} \frac{\tan t}{\sec^{\cancel{4}3} t}\cdot \cancel{\sec t}\cdot \tan t$$

$$= \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} \frac{\tan^2 t}{\sec^3 t}\,dt$$

$$= \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} \tan^2 t \cdot \cos^3 t\,dt = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} \frac{\sin^2 t}{\cancel{\cos^2 t}}\cos^{\cancel{3}} t\,dt$$

$$= \frac{1}{\sqrt{2}-1}\int_0^{\pi/4} \sin^2 t \cos t\,dt$$

$u = \sin t$

$du = \cos t\,dt$

$dt = \frac{1}{\cos t}\,du$

When $t=0$, $u = \sin 0 = 0$

When $t = \pi/4$, $u = \sin \pi/4 = \sqrt{2}/2$

$$= \frac{1}{\sqrt{2}-1}\int_0^{\frac{\sqrt{2}}{2}} u^2 \cdot \cancel{\cos t}\cdot \frac{1}{\cancel{\cos t}}\,du$$

$$= \frac{1}{\sqrt{2}-1}\cdot\left(\frac{u^3}{3}\Big|_0^{\sqrt{2}/2}\right) = \frac{1}{\sqrt{2}-1}\left(\frac{(\sqrt{2}/2)^3}{3} - \frac{0^3}{3}\right)$$

$$= \frac{1}{\sqrt{2}-1}\cdot\frac{1}{6\sqrt{2}} = \boxed{\frac{1}{12-6\sqrt{2}}}$$

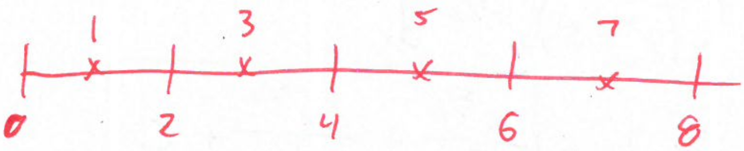
7. Approximate the definite integral $\int_0^8 \frac{1}{1+x^4}$ by using

(a) Left endpoints with $n = 4$

(b) Midpoints with $n = 4$

(c) Simpson's Rule with $n = 4$.

$$\frac{b-a}{n} = \frac{8}{4} = 2$$



(a) $2(f(0)+f(2)+f(4)+f(6))$

$$= \boxed{2\left(\frac{1}{1+0^4}+\frac{1}{1+2^4}+\frac{1}{1+4^4}+\frac{1}{1+6^4}\right)}$$

(b) $2(f(1)+f(3)+f(5)+f(7))$

$$= \boxed{2\left(\frac{1}{1+1^4}+\frac{1}{1+3^4}+\frac{1}{1+5^4}+\frac{1}{1+7^4}\right)}$$

(c) $\frac{2}{3}(f(0)+4f(2)+2f(4)+4f(6)+f(8))$

$$= \boxed{\frac{2}{3}\left(\frac{1}{1+0^4}+\frac{4}{1+2^4}+\frac{2}{1+4^4}+\frac{4}{1+6^4}+\frac{1}{1+8^4}\right)}$$

8. Compute the following indefinite integrals via any method you wish.

(a) $\int \frac{x^2+5}{\sqrt{x}}\, dx$

(b) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

(c) $\int \frac{4x+6}{x^2+4x+5}\, dx$

(a) $\int \frac{x^2+5}{\sqrt{x}}\, dx = \int (x^2+5)x^{-1/2}\, dx$

$= \int x^{3/2} + 5x^{-1/2}\, dx = \boxed{\frac{x^{5/2}}{5/2} + \frac{5x^{1/2}}{1/2} + C}$

(b) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

$u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}\, dx = \frac{1}{2\sqrt{x}}\, dx$

$dx = 2\sqrt{x}\, du$

$\int \frac{e^u}{\cancel{\sqrt{x}}} \cdot 2\cancel{\sqrt{x}}\, du = \int 2e^u\, du = 2e^u + C = \boxed{2e^{\sqrt{x}} + C}$

(c) $x^2+4x+5 = (x+2)^2+1$

$u = x+2 \rightarrow x = u-2$

$du = dx$

$\int \frac{4x+6}{(x+2)^2+1}\, dx$

$= \int \frac{4x+6}{u^2+1}\, du = \int \frac{4(u-2)+6}{u^2+1}\, du = \int \frac{4u-8+6}{u^2+1}\, du$

$= \int \frac{4u-2}{u^2+1}\, du = \int \frac{4u}{u^2+1} - \int \frac{2}{u^2+1}\, du$

$v = u^2+1$

$dv = 2u\, du$

$du = \frac{1}{2u}\, dv$

$= \int \frac{\cancel{4u}^{\,2}}{v} \cdot \frac{1}{\cancel{2u}}\, dv - 2\tan^{-1} u + C$

$= 2\ln|v| - 2\tan^{-1} u + C = 2\ln|u^2+1| - 2\tan^{-1} u + C$

$= \boxed{2\ln|(x+2)^2+1| - 2\tan^{-1}(x+2) + C}$

Trig identities list:

$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$
$\cot^2 x + 1 = \csc^2 x$
$\sin(2x) = 2 \sin x \cos x$
$\cos(2x) = \cos^2 x - \sin^2 x$
$\sin^2(x) = \frac{1-\cos^2 x}{2}$
$\cos^2(x) = \frac{1+\cos^2 x}{2}$