MATH 1953 Exam 1

Name: Solutions

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. (24 pts.) Find the length of the parametric curve $x = t^2 - 2\ln t$, $y = 4t$ from $t = 1$ to $t = 3$.

$$x'(t) = 2t - \frac{2}{t}$$

$$y'(t) = 4$$

$$L = \int_1^3 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_1^3 \sqrt{(2t - \tfrac{2}{t})^2 + (4)^2}\, dt$$

$$= \int_1^3 \sqrt{4t^2 - 8 + \tfrac{4}{t^2} + 16}\, dt$$

$$= \int_1^3 \sqrt{4t^2 + 8 + \tfrac{4}{t^2}}\, dt$$

$$= \int_1^3 \sqrt{(2t + \tfrac{2}{t})^2}\, dt$$

$$= \int_1^3 (2t + \tfrac{2}{t})\, dt$$

$$= t^2 + 2\ln|t| \Big|_1^3$$

$$= (9 + 2\ln(3)) - (1 + 0)$$

$$= \boxed{8 + 2\ln(3).}$$

2. (24 pts.) Find the equation of the tangent line to the curve $r = e^{\theta}$ at $\theta = \pi/2$.

$x = r\cos\theta = e^{\theta}\cos\theta$

$y = r\sin\theta = e^{\theta}\sin\theta$

$\frac{dx}{d\theta} = e^{\theta}(-\sin\theta) + e^{\theta}\cos\theta$

$\frac{dy}{d\theta} = e^{\theta}\cos\theta + e^{\theta}\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{e^{\theta}\cos\theta + e^{\theta}\sin\theta}{-e^{\theta}\sin\theta + e^{\theta}\cos\theta}$$

$$= \frac{\cos\theta + \sin\theta}{-\sin\theta + \cos\theta}$$

Slope: $\frac{dy}{dx}\left(\frac{\pi}{2}\right) = \frac{\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})}{-\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2})}$

$$= \frac{0+1}{-1+0}$$

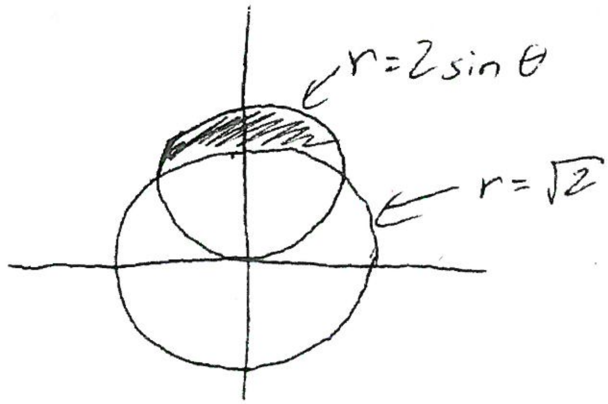
$$= -1.$$

Point: $\underset{\text{polar}}{\underset{(r\ ,\ \theta)}{(e^{\pi/2}, \frac{\pi}{2})}} \longrightarrow \underset{\text{cartesian}}{\underset{(x\ ,\ y)}{(e^{\pi/2}\cos(\frac{\pi}{2}), e^{\pi/2}\sin(\frac{\pi}{2}))}} = (0, e^{\pi/2})$

The equation of the tangent line at $\theta = \frac{\pi}{2}$ is therefore

$y - e^{\pi/2} = -1(x - 0)$ OR $\boxed{y = -x + e^{\pi/2}.}$

3. (25 pts.) Find the area of the region inside the curve $r = 2\sin\theta$ and outside the curve $r = \sqrt{2}$ (see picture below).



Intersection points: $2\sin\theta = \sqrt{2}$

$\Rightarrow \sin\theta = \frac{\sqrt{2}}{2}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$$A = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\tfrac{1}{2}(2\sin\theta)^2 - \tfrac{1}{2}(\sqrt{2})^2\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4\sin^2\theta - 2)\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(4\left(\frac{1-\cos 2\theta}{2}\right) - 2\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} -2\cos 2\theta \, d\theta$$

$$= \frac{1}{2}(-\sin 2\theta)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= -\frac{1}{2}\left[\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right]$$

$$= -\frac{1}{2}(-2)$$

$$= \boxed{1.}$$

4. (9 pts. each) Find the following limits using L'Hospital's Rule:

(a) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$ $\left(\text{type } \frac{0}{0}\right)$

$$\overset{H}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \quad \left(\text{type } \frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x\to 0} \frac{e^x}{2}$$

$$= \boxed{\frac{1}{2}}.$$

(b) $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$ $\left(\text{type } \infty \cdot 0\right)$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\text{type } \frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right) \cdot -x^{-2}}{-x^{-2}}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$$

$$= \boxed{1.}$$

(c) $\lim_{x\to 0+} (1+x)^{\frac{1}{x}}$ $\left(\text{type } 1^\infty\right)$

Let $y = \lim_{x\to 0^+} (1+x)^{1/x}$.

Then $\ln y = \ln\left(\lim_{x\to 0^+} (1+x)^{1/x}\right)$

$$= \lim_{x\to 0^+} \ln\left((1+x)^{1/x}\right)$$

$$= \lim_{x\to 0^+} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \quad \left(\text{type } \frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}$$

$$= 1.$$

Then $\lim_{x\to 0^+} (1+x)^{1/x} = y = e^{\ln y} = e^1 = \boxed{e.}$