MATH 1953 Exam 2

Name: Solutions

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. (10 pts.) If a sequence (x_n) is defined by the recursive formula $x_1 = -2$ and $x_{n+1} = 2x_n + 3$, write the first five terms of the sequence (x_n).

$$x_1 = -2$$

$$x_2 = 2x_1 + 3 = 2(-2) + 3 = -1$$

$$x_3 = 2x_2 + 3 = 2(-1) + 3 = 1$$

$$x_4 = 2x_3 + 3 = 2(1) + 3 = 5$$

$$x_5 = 2x_4 + 3 = 2(5) + 3 = 13$$

2. (25 pts.) Decide whether the improper integral $\int_0^1 2x \ln x \, dx$ converges or diverges, and find its value if it converges.

$$\int_0^1 2x \ln x \, dx = \lim_{t \to 0^+} \int_t^1 2x \ln x \, dx$$

By parts: $u = \ln x$ $\quad dv = 2x\,dx$

$du = \frac{1}{x}dx$ $\quad v = x^2$

$$= \lim_{t \to 0^+} \left[x^2 \ln x \Big|_t^1 - \int_t^1 x^2 \cdot \frac{1}{x} \, dx \right]$$

$$= \lim_{t \to 0^+} \left[x^2 \ln x \Big|_t^1 - \int_t^1 x \, dx \right]$$

$$= \lim_{t \to 0^+} \left[x^2 \ln x \Big|_t^1 - \frac{x^2}{2} \Big|_t^1 \right]$$

$$= \lim_{t \to 0^+} \left[(0 - t^2 \ln t) - \left(\frac{1}{2} - \frac{t^2}{2} \right) \right]$$

$$= \lim_{t \to 0^+} (-t^2 \ln t) - \frac{1}{2}$$

$$= \lim_{t \to 0^+} \left(\frac{\ln t}{-\frac{1}{t^2}} \right) - \frac{1}{2}$$

$$\stackrel{H}{=} \lim_{t \to 0^+} \left(\frac{\frac{1}{t}}{\frac{2}{t^3}} \right) - \frac{1}{2}$$

$$= \lim_{t \to 0^+} \left(\frac{t^3}{2t} \right) - \frac{1}{2}$$

$$= \lim_{t \to 0^+} \left(\frac{t^2}{2} \right) - \frac{1}{2}$$

$$= 0 - \frac{1}{2}$$

$$= -\frac{1}{2}.$$

3. (20 pts.) Use the Comparison Test to decide whether the integral $\int_1^\infty \frac{\sqrt{x}}{x^3+2}\,dx$ converges or diverges (but do NOT attempt to actually evaluate the integral!) You may use facts about convergence/divergence of improper integrals of $\frac{1}{x^p}$ which have been shown in class.

On $(1,\infty)$, $0 \le \frac{\sqrt{x}}{x^3+2} \le \frac{\sqrt{x}}{x^3} = \frac{1}{x^{5/2}}$.

$\int_1^\infty \frac{1}{x^{5/2}}\,dx$ converges since $p=\frac{5}{2}>1$, so by the Comparison Test, $\int_1^\infty \frac{\sqrt{x}}{x^3+2}\,dx$. also converges.

4. (20 pts.) Decide whether the infinite series $\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}}$ converges or diverges, and find its value if it converges. (Hint: it may help to write out the first few terms of the series.)

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{4^n}{5^{n+1}}$$

$$= \frac{4^1}{5^2} + \frac{4^2}{5^3} + \frac{4^3}{5^4} + \frac{4^4}{5^5} + \cdots$$

$$= \frac{4}{5^2}\Big[1 + \underbrace{\left(\frac{4}{5}\right)}_{r} + \frac{4^2}{5^2} + \frac{4^3}{5^3} + \cdots\Big] \quad |r|<1 \Rightarrow \text{conv.}$$

$$= \frac{4}{5^2}\left[\frac{1}{1-\frac{4}{5}}\right]$$

$$= \frac{4}{25} \cdot \frac{5}{1}$$

$$= \frac{4}{5}.$$ The series is convergent and converges to $\frac{4}{5}$.

OR $$\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{4^n}{5^{n+1}}$$

$$= \sum_{n=1}^{\infty} \underbrace{\left(\frac{4}{5^2}\right)}_{a} \underbrace{\left(\frac{4}{5}\right)}_{r}^{n-1} \quad |r|<1 \Rightarrow \text{conv.}$$

$$= \frac{\frac{4}{5^2}}{1-\frac{4}{5}} \quad \frac{a}{1-r}$$

$$= \frac{4}{5}.$$

5. Using any techniques you wish, answer the following questions about the sequence $x_n = \frac{n+2}{2n-1}$.

(a) (5 pts.) Does (x_n) converge to a limit? If so, find the limit.

(x_n) converges to $\frac{1}{2}$:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n+2}{2n-1}$$

$$= \lim_{n\to\infty} \frac{n(1+\frac{2}{n})}{n(2-\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{2-\frac{1}{n}}$$

$$= \frac{1+0}{2-0}$$

$$= \frac{1}{2}.$$

(b) (10 pts.) Is (x_n) increasing, decreasing, or neither?

(x_n) is decreasing:

Let $f(x) = \frac{x+2}{2x-1}$ so that $x_n = f(n)$. Then

$$f'(x) = \frac{(2x-1)\cdot 1 - (x+2)\cdot 2}{(2x-1)^2}$$

$$= \frac{2x-1-2x-4}{(2x-1)^2}$$

$$= \frac{-5}{(2x-1)^2}$$

$$< 0 \quad \text{on } (1,\infty).$$

Since $f(x)$ is decreasing on $(1,\infty)$, then (x_n) is decreasing.

(c) (5 pts.) Is (x_n) bounded from above, bounded from below, both, or neither?

(x_n) is bounded from above and from below since (x_n) is convergent.

(d) (5 pts.) Does the infinite series $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{n+2}{2n-1}$ converge or diverge?

$\sum_{n=1}^{\infty} x_n$ diverges since $\lim_{n\to\infty} x_n \neq 0$.