MATH 1953 Exam 2

Name: Solutions

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn’t take derivatives)

1. (10 pts.) If a sequence \((x_n)\) is defined by the recursive formula \(x_1 = -2\) and \(x_{n+1} = 2x_n + 3\), write the first five terms of the sequence \((x_n)\).

\[
\begin{align*}
x_1 &= -2 \\
x_2 &= 2x_1 + 3 = 2(-2) + 3 = -1 \\
x_3 &= 2x_2 + 3 = 2(-1) + 3 = 1 \\
x_4 &= 2x_3 + 3 = 2(1) + 3 = 5 \\
x_5 &= 2x_4 + 3 = 2(5) + 3 = 13
\end{align*}
\]
2. (25 pts.) Decide whether the improper integral \( \int_{0}^{1} 2x \ln x \, dx \) converges or diverges, and find its value if it converges.

\[
\int_{0}^{1} 2x \ln x \, dx = \lim_{t \to 0^+} \int_{t}^{1} 2x \ln x \, dx
\]

By parts:

\[
u = x^2, \quad \frac{dv}{dx} = 2x, \quad u = \ln x, \quad \frac{du}{dx} = \frac{1}{x}
\]

\[
= \lim_{t \to 0^+} \left[ x^2 \ln x \bigg|_{t}^{1} - \int_{t}^{1} x^2 \frac{1}{x} \, dx \right]
\]

\[
= \lim_{t \to 0^+} \left[ x^2 \ln x \bigg|_{t}^{1} - \int_{t}^{1} x \, dx \right]
\]

\[
= \lim_{t \to 0^+} \left[ x^2 \ln x \bigg|_{t}^{1} - \frac{x^2}{2} \bigg|_{t}^{1} \right]
\]

\[
= \lim_{t \to 0^+} \left[ (1 - t^2 \ln t) - \left( \frac{1}{2} - \frac{t^2}{2} \right) \right]
\]

\[
= \lim_{t \to 0^+} (-t^2 \ln t) - \frac{1}{2}
\]

\[
= \lim_{t \to 0^+} \left( \frac{t \ln t}{-\frac{1}{2}} \right) - \frac{1}{2}
\]

\[
= \lim_{t \to 0^+} \left( \frac{t}{\frac{1}{2}} \right) - \frac{1}{2}
\]

\[
= \lim_{t \to 0^+} \left( \frac{t^3}{2t} \right) - \frac{1}{2}
\]

\[
= \lim_{t \to 0^+} \left( \frac{t^2}{2} \right) - \frac{1}{2}
\]

\[
= 0 - \frac{1}{2}
\]

\[
= -\frac{1}{2}
\]
3. (20 pts.) Use the Comparison Test to decide whether the integral \( \int_{1}^{\infty} \frac{\sqrt{x}}{x^3 + 2} \, dx \) converges or diverges (but do NOT attempt to actually evaluate the integral!) You may use facts about convergence/divergence of improper integrals of \( \frac{1}{x^p} \) which have been shown in class.

On \((1, \infty)\), \(0 \leq \frac{\sqrt{x}}{x^3 + 2} \leq \frac{\sqrt{x}}{x^3} = \frac{1}{x^{5/2}}.\)

\(\int_{1}^{\infty} \frac{1}{x^{5/2}} \, dx \) converges since \(p = \frac{5}{2} > 1\), so by the Comparison Test, \(\int_{1}^{\infty} \frac{\sqrt{x}}{x^3 + 2} \, dx\) also converges.
4. (20 pts.) Decide whether the infinite series \( \sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}} \) converges or diverges, and find its value if it converges. (Hint: it may help to write out the first few terms of the series.)

\[
\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{4^n}{5^{n+1}}
\]

\[
= \frac{4^1}{5^2} + \frac{4^2}{5^3} + \frac{4^3}{5^4} + \frac{4^4}{5^5} + \ldots
\]

\[
= \frac{4}{5^2} \left[ 1 + \left( \frac{4}{5} \right) + \left( \frac{4}{5} \right)^2 + \left( \frac{4}{5} \right)^3 + \ldots \right] \quad \text{if} \quad |r| < 1 \Rightarrow \text{conv.}
\]

\[
= \frac{4}{5^2} \left[ \frac{1}{1 - \frac{4}{5}} \right]
\]

\[
= \frac{4}{25} \cdot \frac{5}{1}
\]

\[
= \frac{4}{5} \quad \text{The series is convergent and converges to} \quad \frac{4}{5}.
\]

OR

\[
\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{4^n}{5^{n+1}}
\]

\[
= \sum_{n=1}^{\infty} \frac{\left( \frac{4}{5} \right)^n}{a} \quad \text{if} \quad |r| < 1 \Rightarrow \text{conv.}
\]

\[
= \frac{\frac{4}{5^2}}{1 - \frac{4}{5}} \quad \frac{a}{1-r}
\]

\[
= \frac{4}{5}.
\]
5. Using any techniques you wish, answer the following questions about the sequence \( x_n = \frac{n+2}{2n-1} \).

(a) (5 pts.) Does \((x_n)\) converge to a limit? If so, find the limit.

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{n+2}{2n-1} \\
= \lim_{n \to \infty} \frac{n(1 + \frac{2}{n})}{n(2 - \frac{1}{n})} \\
= \lim_{n \to \infty} \frac{1 + \frac{2}{n}}{2 - \frac{1}{n}} \\
= \frac{1+0}{2-0} \\
= \frac{1}{2}.
\]

(b) (10 pts.) Is \((x_n)\) increasing, decreasing, or neither?

\((x_n)\) is decreasing:

Let \( f(x) = \frac{x+2}{2x-1} \) so that \( x_n = f(n) \). Then

\[
f'(x) = \frac{(2x-1) \cdot 1 - (x+2) \cdot 2}{(2x-1)^2} \\
= \frac{2x-1 - 2x - 4}{(2x-1)^2} \\
= \frac{-5}{(2x-1)^2} \\
\leq 0 \quad \text{on} \quad (1, \infty).
\]

Since \( f(x) \) is decreasing on \((1, \infty)\), then \((x_n)\) is decreasing.
(c) (5 pts.) Is \( (x_n) \) bounded from above, bounded from below, both, or neither?

\( (x_n) \) is bounded from above and from below since \( (x_n) \) is convergent.

(d) (5 pts.) Does the infinite series \( \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{n + 2}{2n - 1} \) converge or diverge?

\[ \sum_{n=1}^{\infty} x_n \text{ diverges since } \lim_{n \to \infty} x_n \neq 0. \]