MATH 1953 Midterm 3

Name: Solutions

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You DO NOT need to simplify your answers unless otherwise indicated! You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. (20 pts.) Find the exact value of the convergent telescoping infinite series $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{n+3}-\frac{1}{n}\right)$.

$$\sum_{n=3}^{\infty}\left(\frac{1}{n+3}-\frac{1}{n}\right) = \left(\cancel{\frac{1}{6}}-\frac{1}{3}\right)+\left(\cancel{\frac{1}{7}}-\frac{1}{4}\right)+\left(\cancel{\frac{1}{8}}-\frac{1}{5}\right)+\left(\cancel{\frac{1}{9}}-\cancel{\frac{1}{6}}\right)+\cdots$$

$$= -\frac{1}{3}-\frac{1}{4}-\frac{1}{5}$$

$$= \frac{-47}{60}.$$

2. (25 pts.) Use the Alternating Series Test to decide whether the infinite series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{e^n}$$

converges or diverges. DO NOT attempt to find the exact value of the series, but DO explain why any hypotheses you need for the test are true!

- $x_n = \frac{n}{e^n} \geq 0$ for all $n \geq 1$

- $\lim_{n\to\infty} \frac{n}{e^n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{e^n}$

$$= 0.$$

- let $f(x) = \frac{x}{e^x}$. Then

$$f'(x) = \frac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2}$$

$$= \frac{e^x(1-x)}{(e^x)^2}$$

$$= \frac{1-x}{e^x}$$

$$\leq 0 \text{ on } (1, \infty).$$

$f(x)$ is decreasing on $(1,\infty)$, so x_n is decreasing.

By the Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n \frac{n}{e^n}$ converges.

3. (25 pts.) Using either the Comparison Test or Limit Comparison Test (at least one of them will work for this problem), decide whether the infinite series

$$\sum_{n=1}^{\infty} \frac{n^2+3}{n^4+n}$$

converges or diverges. DO NOT attempt to find the exact value of the series, but DO explain why any hypotheses you need for the test are true!

We will use LCT. Let $a_n = \frac{n^2+3}{n^4+n}$ and $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+3}{n^4+n}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^2+3}{n^4+n} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^4+3n^2}{n^4+n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{3}{n^2}}{1+\frac{1}{n^3}}$$

$$= \frac{1+0}{1+0}$$

$$= 1.$$

Since $a_n, b_n \geq 0$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive and finite, then by LCT $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2 > 1$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2+3}{n^4+n}$ converges.

4. (a) (25 pts.) Use the Integral Test to decide whether the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges or diverges. DO NOT attempt to find the exact value of the series, but DO explain why any hypotheses you need for the test are true!

Let $f(x) = \frac{1}{x(\ln x)^2}$, which is continuous and positive on $[2, \infty)$.
$f(x)$ is decreasing since the denominator $x(\ln x)^2$ is increasing.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx \qquad u = \ln x, \quad du = \frac{1}{x}dx$$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2}\,du$$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u^{-2}\,du$$

$$= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{\ln t} - \left(-\frac{1}{\ln 2}\right)\right]$$

$$= 0 + \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2}.$$

Since the improper integral $\int_2^{\infty} \frac{1}{x(\ln x)^2}dx$ converges, then by the Integral Test the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

(b) (5 pts.) Decide how large N needs to be so that the sum of the first N terms definitely approximates the infinite series from (a) to within $0.1 = \frac{1}{10}$. (If you don't have a calculator, you don't need to simplify your answer, and I won't deduct points for not rounding to an integer!)

We need to find the least value of N such that

$$\int_N^\infty \frac{1}{x(\ln x)^2}\,dx < \frac{1}{10}.$$

From (a) we know that $\int_N^\infty \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln N}$.

Then $\int_N^\infty \frac{1}{x(\ln x)^2}\,dx < \frac{1}{10}$

$$\iff \frac{1}{\ln N} < \frac{1}{10}$$

$$\iff 10 < \ln N$$

$$\iff e^{10} < e^{\ln N}$$

$$\iff e^{10} < N.$$

$e^{10} \approx 22026.5$, so N needs to be at least 22027.