MATH 1953 Midterm 4

Name: Solution.

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You DO NOT need to simplify your answers unless otherwise indicated! You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. (12 pts. each) For each of the following series, use any of the convergence/divergence tests from our class to decide whether the series converges or diverges. DO NOT attempt to find the exact value of the series, but DO explain why any hypotheses you need for your chosen test are true!

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^5+n+1}{n^7-n^4}$

We will use LCT. Let $a_n = \frac{n^5+n+1}{n^7-n^4}$ and $b_n = \frac{1}{n^2}$.

Then $a_n, b_n \geq 0$ for $n \geq 2$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^5+n+1}{n^7-n^4}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^5+n+1}{n^7-n^4} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^7+n^3+n^2}{n^7-n^4}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^4}+\frac{1}{n^5}}{1-\frac{1}{n^3}}$$

$$= \frac{1+0+0}{1-0}$$

$$= 1,$$

which is positive and finite. By LCT $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} b_n$ have the same convergence. Since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p=2>1$, then $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{n^5+n+1}{n^7-n^4}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3 5^n}$

We will use the Comparison Test.

$$0 \leq \frac{1}{n^3 5^n} < \frac{1}{5^n} \text{ for } n \geq 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges as a geometric series with $r = \frac{1}{5}$,

then $\sum_{n=1}^{\infty} \frac{1}{n^3 5^n}$ converges.

(c) $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right)$

The series diverges by the Divergence Test.

Since $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0$ then $\lim_{n\to\infty} (-1)^n \cos\left(\frac{1}{n}\right) \neq 0$.

(d) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

We'll use the Comparison Test.

$$0 \leq \frac{e^{\frac{1}{n}}}{n^2} \leq \frac{e}{n^2} \text{ for } n \geq 1.$$

Since $\sum_{n=1}^{\infty} \frac{e}{n^2} = e \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p=2$,
then $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ converges by the Comparison Test.

2. (32 pts.) Find the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n4^n}$.

We will use the Root Test.

$$\lim_{n\to\infty} \left|\frac{(2x-6)^n}{n4^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{2x-6}{\sqrt[n]{n}\cdot 4}\right|$$

$$= \frac{|2x-6|}{4}$$

$$= \frac{|x-3|}{2}.$$

By the Root Test the series converges when $\frac{|x-3|}{2} < 1 \iff |x-3| < 2$

$\iff -2 < x-3 < 2 \iff 1 < x < 5.$

$\underline{x=1}$: $\sum_{n=1}^{\infty} \frac{(2\cdot 1-6)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$

$\frac{1}{n} \geq 0$ for $n \geq 1$, $\lim_{n\to\infty} \frac{1}{n} = 0$, and $x_n = \frac{1}{n}$ is decreasing since the denominator is increasing. By AST, the series converges. So $x=1$ is included in the interval of convergence.

$\underline{x=5}$: $\sum_{n=1}^{\infty} \frac{(2\cdot 5-6)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$

This series (harmonic series) diverges as a ~~p~~ p-series with $p=1$. So $x=5$ is not included in the interval of convergence.

The interval of convergence is $[1,5)$.

3. (5 pts. each) Answer each of the following questions about a mystery power series $\sum_{n=0}^{\infty} c_n(x-2)^n$ True or False. You DO NOT have to explain your answers.

T **F** The series $\sum_{n=0}^{\infty} c_n(x-2)^n$ must converge at its center $x = 2$.

T **F** It is possible for the series $\sum_{n=0}^{\infty} c_n(x-2)^n$ to converge at every real number.

T **F** It is possible for the series $\sum_{n=0}^{\infty} c_n(x-2)^n$ to converge at $x = 3$ and $x = 5$, but diverge at $x = 4$.

T **F** It is possible for the series $\sum_{n=0}^{\infty} c_n(x-2)^n$ to converge at every positive number, but diverge at every negative number.