MATH 1953 Practice Midterm 1

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn’t take derivatives)

1. Find the equations of both tangent lines to the parametric curve \( x = t^2, \) \( y = 2t^5 - 8t^3 \) at the intersection point (4, 0).

Solution: First we need a formula for \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{10t^4 - 24t^2}{2t} = 5t^3 - 12t.
\]

To plug in, we need \( t \)-values, but we were given the \((x, y)\) point (4, 0). To solve for \( t \)-values, we use \( x \): if \( x = 4 \), then \( t^2 = 4 \), meaning \( t = \pm 2 \). If we plug those into \( y = 2t^5 - 8t^3 \), we will indeed get 0 each time, meaning that (4, 0) does correspond to the two \( t \)-values \( t = -2 \) and \( t = 2 \). Then, we plug into \( \frac{dy}{dx} \).

For \( t = 2 \), \( \frac{dy}{dx} = 5(2^3) - 12(2) = 16 \), so our slope is 16. We then plug \( m = 16 \) and \((4, 0)\) into point-slope to get \( y - 0 = 16(x - 4) \), or \( y = 16x - 64 \).

For \( t = -2 \), \( \frac{dy}{dx} = 5(-2^3) - 12(-2) = -16 \), so our slope is \(-16\). We then plug \( m = -16 \) and \((4, 0)\) into point-slope to get \( y - 0 = -16(x - 4) \), or \( y = -16x + 64 \).
2. Find the area of the region inside the curve $r = 2 \cos \theta$ and outside the curve $r = 1$ (see picture below).

Solution: We first need values of $\theta$ at the intersection points to know our limits of integration. To find this, we set the curves equal:

$2 \cos \theta = 1 \implies \cos \theta = 1/2 \implies \theta = \pm \pi/3$.

From the picture, we see that our area occurs between $\theta = -\pi/3$ and $\theta = \pi/3$. In this region, $2 \cos \theta$ is the outer function and $r = 1$ is the inner, so our integral is

\[
\int_{-\pi/3}^{\pi/3} \frac{1}{2} (2 \cos \theta)^2 - \frac{1}{2} (1^2) \, d\theta = \int_{-\pi/3}^{\pi/3} 2 \cos^2 \theta - \frac{1}{2} \, d\theta = \int_{-\pi/3}^{\pi/3} 1 + \cos 2\theta - \frac{1}{2} \, d\theta =
\]

\[
\int_{-\pi/3}^{\pi/3} \frac{1}{2} \cos 2\theta \, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{2} \bigg|_{-\pi/3}^{\pi/3} = \frac{\pi}{6} + \frac{\sin(2\pi/3)}{2} - \frac{-\pi}{6} - \frac{\sin(-2\pi/3)}{2} = \frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{6} + \frac{\sqrt{3}}{2} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.
\]
3. Find the length of the polar curve \( r = \sin \theta + \cos \theta \) between \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \).

**Solution:** The formula for polar length is

\[
\int_{0}^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_{0}^{\pi/2} \sqrt{(\sin \theta + \cos \theta)^2 + (\cos \theta - \sin \theta)^2} \, d\theta = \\
\int_{0}^{\pi/2} \sqrt{\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta + \sin^2 \theta} \, d\theta = \\
\int_{0}^{\pi/2} \sqrt{2 \sin^2 \theta + 2 \cos^2 \theta} \, d\theta = \int_{0}^{\pi/2} \sqrt{2} \, d\theta = \sqrt{2} \theta \bigg|_{0}^{\pi/2} = \frac{\sqrt{2} \pi}{2}.
\]
4. Find the following limits using L’Hospital’s Rule:

(a) \( \lim_{x \to 1} \frac{x^3 - 1}{x^5 - 1} \)

**Solution:** Top and bottom both go to 0, so we use L’Hospital’s:

\[
\lim_{x \to 1} \frac{x^3 - 1}{x^5 - 1} = \lim_{x \to 1} \frac{3x^2}{5x^4} = \frac{3}{5}.
\]

(b) \( \lim_{x \to -\infty} x^2 e^x \)

**Solution:** We need to rewrite as a fraction. It’s better to put \( x^2 \) on the top than \( x^{-2} \) on the bottom, so we rewrite as \( \lim_{x \to -\infty} \frac{x^2}{e^x} \). The top and bottom both go to \( \infty \), so we use L’Hospital’s:

\[
\lim_{x \to -\infty} \frac{x^2}{e^x} = \lim_{x \to -\infty} \frac{2x}{-e^x}.
\]

The top and bottom still both go to \( \infty \), so we use L’Hospital’s again:

\[
\lim_{x \to -\infty} \frac{2x}{-e^x} = \lim_{x \to -\infty} \frac{2}{-e^x}.
\]

Now the top is 2 and the bottom goes to \( \infty \), so the limit is 0.

(c) \( \lim_{x \to \infty} x^{1/x} \)

**Solution:** This is an exponential, so we take a natural log, evaluate the limit, then take \( e \) to the power of the answer to “undo the ln.”

\[
\lim_{x \to \infty} \ln(x^{1/x}) = \lim_{x \to \infty} \frac{1}{x} \ln x = \lim_{x \to \infty} \frac{\ln x}{x}.
\]

The top and bottom both go to \( \infty \), so we use L’Hospital’s:

\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0.
\]

We then take \( e \) to the power of this answer to find the original limit, which is \( e^0 = 1 \).