MATH 1953 Practice Midterm 2 Solutions

Name: ____

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. Does the improper integral $\int_1^\infty xe^{-x}\ dx$ converge or diverge?

Solution:

$$\int_1^\infty xe^{-x}\ dx = \lim_{t\to\infty}\int_1^t xe^{-x}\ dx.$$

We need to use integration by parts with $u = x$ and $dv = e^{-x}\ dx$, meaning that $du = 1\ dx$ and $v = -e^{-x}$. So,

$$\int_1^t xe^{-x}\ dx = \int_1^t udv = uv|_1^t - \int_1^t vdu = -xe^{-x}|_1^t - \int_1^t (-e^{-x})\ dx = -xe^{-x}|_1^t - (e^{-x})|_1^t$$

$$= -te^{-t} - (-e^{-1}) - (e^{-t} - e^{-1}) = -te^{-t} + \frac{1}{e} - e^{-t} + \frac{1}{e}.$$

Now we need the limit as $t \to \infty$. Clearly $e^{-t} = \frac{1}{e^t}$ approaches 0 as $t \to \infty$, but $\lim_{t\to\infty} te^{-t}$ is an indeterminate form. So, we use L'Hospital's:

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty}\frac{t}{e^t} = \lim_{t\to\infty}\frac{1}{e^t} = 0.$$

So, our original improper integral is

$$\lim_{t\to\infty} -te^{-t} + \frac{1}{e} - e^{-t} + \frac{1}{e} = -0 + \frac{1}{e} - 0 + \frac{1}{e} = \frac{2}{e}.$$

2. Use the Comparison Test to decide whether the improper integral

$$\int_1^\infty \frac{2x^2}{x^5+10}\,dx$$

converges or diverges. DO NOT attempt to find the value of the integral!

Solution: We notice that $\frac{2x^2}{x^5+10} \leq \frac{2x^2}{x^5} = \frac{2}{x^3}$. Also,

$$\int_1^\infty \frac{2}{x^3}\,dx = 2\int_1^\infty \frac{1}{x^3}\,dx$$

converges since the exponent is greater than 1. (Showed in class!) Finally, all of these functions are positive for x between 1 and ∞.

So, we can use the Comparison Test: since $\frac{2x^2}{x^5+10} \leq \frac{2}{x^3}$, both are positive, and $\displaystyle\int_1^\infty \frac{2}{x^3}\,dx$ converges, the original integral $\displaystyle\int_1^\infty \frac{2x^2}{x^5+10}\,dx$ converges as well.

3. Using any techniques you wish, answer the following questions about the sequence $x_n = \frac{n}{n^2+1}$.

(a) Does (x_n) converge to a limit? If so, find the limit.

Solution: There are several solutions. We did one with L'Hospital's in class, but you could also use algebra:

$$\frac{n}{n^2+1} = \frac{n \cdot \frac{1}{n^2}}{(n^2+1)\frac{1}{n^2}} = \frac{1/n}{1+\frac{1}{n^2}}.$$

Since $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$, the whole sequence approaches $\frac{0}{1+0} = 0$.

(b) Is (x_n) increasing, decreasing, or neither?

Solution: Again there are several solutions. We did a caclulus solution in class, but you could also compare adjacent terms:

$$x_n \geq x_{n+1} \longleftrightarrow \frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1} \longleftrightarrow \frac{n}{n^2+1} \geq \frac{n+1}{n^2+2n+2}$$

$$\longleftrightarrow n(n^2+2n+2) \geq (n+1)(n^2+1) \longleftrightarrow n^3+2n^2+2n \geq n^3+n^2+n+1 \longleftrightarrow n^2+n \geq 1,$$

which is true since $n \geq 1$ for any sequence. So, $x_n \geq x_{n+1}$ is always true, i.e. x_n is decreasing.

(c) Is (x_n) bounded from above, bounded from below, both, or neither?

Solution: Since (x_n) is convergent (from (a)), it is automatically bounded from above and below. (This was shown in class!)

4. Determine whether the following series converges or diverges, and if it converges, find its sum exactly.

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^{n-1}}$$

Solution: We notice that this seems to be a geometric series, since the top and bottom each come from powers of a single number. So, we write the first few terms:

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^{n-1}} = \frac{3^2}{5^0}+\frac{3^3}{5^1}+\frac{3^4}{5^2}+\ldots = \frac{3^2}{5^0}\left(1+\frac{3^1}{5^1}+\frac{3^2}{5^2}+\ldots\right) = \frac{3^2}{5^0}\left(1+\frac{3}{5}+\left(\frac{3}{5}\right)^2+\ldots\right) =$$

The series inside the parentheses is geometric with $r = \frac{3}{5}$, which is in $(-1, 1)$. So, we use the formula $\frac{1}{1-r}$ for the value of a convergent geometric series, and our answer is

$$\frac{3^2}{5^0}\frac{1}{1-\frac{3}{5}} = 9\frac{1}{\frac{2}{5}} = \frac{45}{2} = 22.5.$$

ALTERNATE SOLUTION: if you don't want to write out terms, you can just rewrite as

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^{n-1}} = \sum_{n=1}^{\infty} 3^2 \cdot \frac{3^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} 3^2 \left(\frac{3}{5}\right)^{n-1} = 3^2\frac{1}{1-\frac{3}{5}},$$

giving the same answer.