MATH 1953 Practice Midterm 3 Solutions

Name: ____

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. Find the value of the convergent infinite series $\sum_{n=3}^{\infty} \frac{4}{n^2+6n+8}$ by using partial fractions to rewrite it as a telescoping series.

Solution: We can rewrite

$$\frac{4}{n^2+6n+8} = \frac{4}{(n+2)(n+4)} = \frac{A}{n+2} + \frac{B}{n+4},$$

yielding

$$4 = A(n+4) + B(n+2).$$

Plugging in $n = -4$ yields $4 = -2B$, so $B = -2$. Plugging in $n = -2$ yields $4 = 2A$, so $A = 2$. So,

$$\sum_{n=3}^{\infty} \frac{4}{n^2+6n+8} = \sum_{n=3}^{\infty} \frac{2}{n+2} - \frac{2}{n+4} =$$

$$\frac{2}{5} - \frac{2}{7} + \frac{2}{6} - \frac{2}{8} + \frac{2}{7} - \frac{2}{9} + \frac{2}{8} - \frac{2}{10} \ldots = \frac{2}{5} + \frac{2}{6}.$$

2. Use the Integral Test to decide whether the infinite series

$$\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$$

converges or diverges. DO NOT attempt to find the exact value of the series!

Solution: To use the Integral Test, we need to confirm that the terms $x_n = \frac{2n}{n^2+1}$ are positive and decreasing. Positivity is obvious since $n > 1$. To check decreasing, we represent x_n as a function of x and take a derivative:

$$\left(\frac{2x}{x^2+1}\right)' = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}.$$

This derivative is negative since $x \geq 1$ for all terms in our sum, and so the terms x_n are decreasing.

Therefore, the convergence status of $\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$ is the same as that of the improper integral $\int_1^{\infty} \frac{2x}{x^2+1}\, dx$, and so we simply need to decide whether that integral converges.

$$\int_1^{\infty} \frac{2x}{x^2+1}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{2x}{x^2+1}\, dx = \lim_{t\to\infty} \ln(x^2+1)|_1^t = \lim_{t\to\infty} \ln(t^2+1) - \ln 2.$$

(the integral was obtained with a basic u-substitution, where $u = x^2+1$ and $du = 2x\, dx$.)

Since $t \to \infty$, $t^2+1 \to \infty$, and so $\ln(t^2+1) \to \infty$ as well. Therefore, this integral diverges, meaning that the original series diverges as well by the Integral Test.

3. Using either the Comparison Test or Limit Comparison Test, decide whether the infinite series

$$\sum_{n=1}^{\infty} \frac{3^n - 2}{4^n + 5}$$

converges or diverges. DO NOT attempt to find the exact value of the series!

Solution: We'll use the Limit Comparison Test with $x_n = \frac{3^n-2}{4^n+5}$ (the original series) and $y_n = \frac{3^n}{4^n}$. Then,

$$\frac{x_n}{y_n} = \frac{(3^n - 2)/(4^n + 5)}{(3^n)/(4^n)} = \frac{(3^n - 2)\frac{1}{3^n}}{(4^n + 5)\frac{1}{4^n}} = \frac{1 - \frac{2}{3^n}}{1 + \frac{5}{4^n}}.$$

This means that $\frac{x_n}{y_n} \to 1$, and since this limit is a nonzero number, we can use the Limit Comparison Test. This means that the convergence status of the original series is the same as the status of

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n.$$

This series is geometric with $r = \frac{3}{4}$, which is between -1 and 1, and so it converges. Therefore, the original series also converges by the Limit Comparison Test.

4. (a) Use the Alternating Series Test to decide whether the infinite series

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$$

converges or diverges. DO NOT attempt to find the exact value of the series!

Solution: The Alternating Series Test requires us to take x_n to be the part of the series without the alternating sign, i.e. $x_n = \frac{1}{n!}$. As $n \to \infty$, $n! \to \infty$ as well (you can see this since $n! \geq n$.) So, $x_n = \frac{1}{n!} \to 0$.

Similarly, $n!$ is increasing (because $(n+1)! = (n+1)n! > n!$), so $x_n = \frac{1}{n!}$ is decreasing. Since $x_n \to 0$ and is decreasing, we can use the Alternating Series Test, and so the series $\sum_{n=1}^{\infty}(-1)^n x_n = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$ converges.

(b) Give a partial sum which approximates the infinite series from (a) to within $\frac{1}{100}$. (A basic calculator may be useful for this!)

Solution: Remember that the error/distance between the infinite series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$ and the partial sum of the first N terms is less than x_{N+1}. So, we just need to make N big enough so that $x_{N+1} < \frac{1}{100}$. We can see that

$$x_1 = \frac{1}{1!} = 1,\ x_2 = \frac{1}{2!} = \frac{1}{2},\ x_3 = \frac{1}{1!} = \frac{1}{6},\ x_4 = \frac{1}{1!} = \frac{1}{24},\ x_5 = \frac{1}{1!} = \frac{1}{120}.$$

Since $x_5 < \frac{1}{100}$, we can take $N+1 = 5$, i.e. $N = 4$. So, the partial sum from the first 4 terms is within $\frac{1}{100}$ of the infinite series. Therefore, our approximation is

$$-\frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}.$$

You wouldn't need to simplify, but least common denominator can be used to reduce this to $\frac{-5}{8}$.