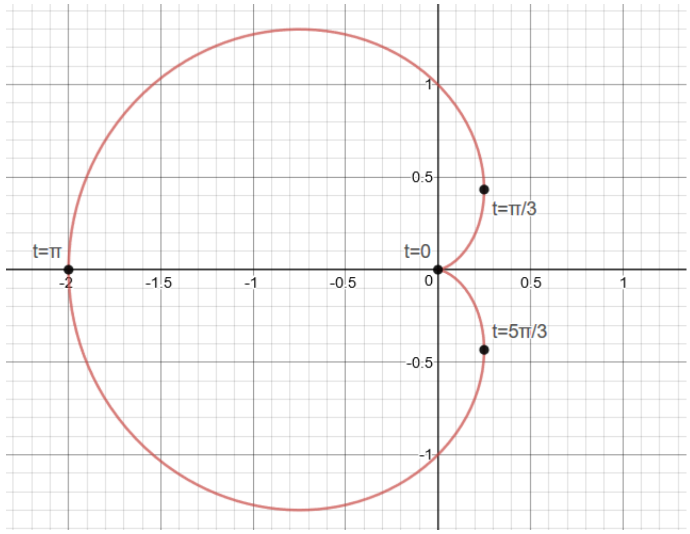
Friday Week 1 - Solutions

Calculus III



Above is a graph of a cardioid, parametrized by

$$x(t) = (1 - \cos t)\cos t$$
$$y(t) = (1 - \cos t)\sin t$$
$$0 \le t \le 2\pi$$

(a) Use the graph to determine when $x(t)$ is increasing/decreasing, and when $y(t)$ positive/negative.

$x(t)$ increasing: $[0, \frac{\pi}{3}] \cup [\pi, \frac{5\pi}{2}]$

$x(t)$ decreasing: $[\frac{\pi}{3}, \pi] \cup [\frac{5\pi}{3}, 2\pi]$

$y(t) \ge 0$: $[0, \pi]$

$y(t) \le 0$: $[\pi, 2\pi]$

(b) Give an expression for the area enclosed by the cardioid (but do not integrate). You may make use of symmetry.

$x(t) = \cos t - \cos^2 t$, and $x'(t) = -\sin t + 2\sin t\cos t$. From (a) we have

$[0, \frac{\pi}{3}]$: $x(t)$ is inc. and $y(t) \ge 0$

$[\frac{\pi}{3}, \pi]$: $x(t)$ is dec. and $y(t) \ge 0$

$[\pi, \frac{5\pi}{3}]$: $x(t)$ is inc. and $y(t) \le 0$

$[\frac{5\pi}{3}, 2\pi]$: $x(t)$ is dec. and $y(t) \le 0$

So the area enclosed by the cardioid is

$$\int_0^{\frac{\pi}{3}} (1 - \cos t)\sin t(-\sin t + 2\sin t\cos t)\, dt$$

$$+ \int_\pi^{\frac{\pi}{3}} (1 - \cos t)\sin t(-\sin t + 2\sin t\cos t)\, dt$$

$$- \int_\pi^{\frac{5\pi}{3}} (1 - \cos t)\sin t(-\sin t + 2\sin t\cos t)\, dt$$

$$- \int_{2\pi}^{\frac{5\pi}{2}} (1 - \cos t)\sin t(-\sin t + 2\sin t\cos t)\, dt$$

$$= \int_0^{\frac{\pi}{3}} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt$$
$$- \int_{\frac{\pi}{3}}^{\pi} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt$$
$$- \int_{\pi}^{\frac{5\pi}{3}} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt$$
$$+ \int_{\frac{5\pi}{2}}^{2\pi} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt.$$

Or, using symmetry, on $[0, \pi]$ the area enclosed by the curve and the x-axis is

$$\int_0^{\frac{\pi}{3}} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt$$
$$+ \int_{\pi}^{\frac{\pi}{3}} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt$$
$$= \int_0^{\frac{\pi}{3}} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt$$
$$- \int_{\frac{\pi}{3}}^{\pi} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt$$

So the total area is

$$2\left[\int_0^{\frac{\pi}{3}} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt - \int_{\frac{\pi}{3}}^{\pi} (1-\cos t)\sin t(-\sin t + 2\sin t\cos t)\ dt\right].$$

Consider the parametric curve

$$x(t) = t^3 - 4t$$
$$y(t) = t^2$$
$$0 \leq t \leq 2$$

(a) Without graphing, determine when $x(t)$ is increasing/decreasing, and when $y(t)$ is positive/negative.

We'll use the first derivative test to determine when $x(t)$ is increasing or decreasing.

$x'(t) = 0 \iff 3t^2 - 4 = 0 \iff t = \pm\frac{2\sqrt{3}}{3}$

Only $t = \frac{2\sqrt{3}}{3}$ is in the interval $[0, 2]$, so using test points $t = 0$ and $t = 2$ we have

$x'(0) = -4 \leq 0 \implies x(t)$ is decreasing on $[0, \frac{2\sqrt{3}}{3}]$

$x'(2) = 8 \geq 0 \implies x(t)$ is increasing on $[\frac{2\sqrt{3}}{3}, 2]$

Since $y(t) = t^2$ then $y(t) \geq 0$ on $[0, 2]$.

(b) Give an expression for the area enclosed by the curve and the y-axis (but do not integrate).

$[0, \frac{2\sqrt{3}}{3}]$: $x(t)$ is decreasing and $y(t) \geq 0$

$[\frac{2\sqrt{3}}{3}, 2]$: $x(t)$ is increasing and $y(t) \geq 0$

Then the area enclosed by the curve and the y-axis is

$$\int_{\frac{2\sqrt{3}}{3}}^{0} t^2(3t^2 - 4)\,dt + \int_{\frac{2\sqrt{3}}{3}}^{2} t^2(3t^2 - 4)\,dt = -\int_{0}^{\frac{2\sqrt{3}}{3}} t^2(3t^2 - 4)\,dt + \int_{\frac{2\sqrt{3}}{3}}^{2} t^2(3t^2 - 4)\,dt$$