Above is a graph of a cardioid, parametrized by

\[ x(t) = (1 - \cos t) \cos t \]
\[ y(t) = (1 - \cos t) \sin t \]
\[ 0 \leq t \leq 2\pi \]

(a) Use the graph to determine when \( x(t) \) is increasing/decreasing, and when \( y(t) \) positive/negative.

\[ x(t) \] increasing: \([0, \pi/3] \cup [\pi, 5\pi/3]\] \[ y(t) \geq 0: [0, \pi] \]
\[ x(t) \] decreasing: \([\pi/3, \pi] \cup [5\pi/3, 2\pi]\] \[ y(t) \leq 0: [\pi, 2\pi] \]

(b) Give an expression for the area enclosed by the cardioid (but do not integrate). You may make use of symmetry.

\[ x(t) = \cos t - \cos^2 t, \text{ and } x'(t) = -\sin t + 2 \sin t \cos t. \text{ From (a) we have} \]
\[ [0, \pi/3]: x(t) \text{ is inc. and } y(t) \geq 0 \]
\[ [\pi/3, \pi]: x(t) \text{ is dec. and } y(t) \geq 0 \]
\[ [\pi, 5\pi/3]: x(t) \text{ is inc. and } y(t) \leq 0 \]
\[ [5\pi/3, 2\pi]: x(t) \text{ is dec. and } y(t) \leq 0 \]

So the area enclosed by the cardioid is

\[ \int_{0}^{\pi/3} (1 - \cos t) \sin t (-\sin t + 2 \sin t \cos t) \, dt \]
\[ + \int_{\pi/3}^{\pi} (1 - \cos t) \sin t (-\sin t + 2 \sin t \cos t) \, dt \]
\[ - \int_{\pi}^{5\pi/3} (1 - \cos t) \sin t (-\sin t + 2 \sin t \cos t) \, dt \]
\[ - \int_{5\pi/3}^{2\pi} (1 - \cos t) \sin t (-\sin t + 2 \sin t \cos t) \, dt \]
\[
\int_{0}^{\pi/3} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt \\
- \int_{\pi/3}^{\pi} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt \\
- \int_{\pi}^{5\pi/3} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt \\
+ \int_{5\pi/3}^{2\pi} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt.
\]

Or, using symmetry, on \([0, \pi]\) the area enclosed by the curve and the \(x\)-axis is

\[
\int_{0}^{\pi/3} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt \\
+ \int_{\pi/3}^{\pi} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt \\
= \int_{0}^{\pi/3} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt \\
- \int_{\pi/3}^{\pi} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt
\]

So the total area is

\[
2 \left[ \int_{0}^{\pi/3} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt - \int_{\pi/3}^{\pi} (1 - \cos t) \sin t(-\sin t + 2\sin t \cos t) \, dt \right].
\]
Consider the parametric curve

\[ x(t) = t^3 - 4t \]
\[ y(t) = t^2 \]
\[ 0 \leq t \leq 2 \]

(a) Without graphing, determine when \( x(t) \) is increasing/decreasing, and when \( y(t) \) is positive/negative.

We’ll use the first derivative test to determine when \( x(t) \) is increasing or decreasing.

\[ x'(t) = 0 \iff 3t^2 - 4 = 0 \iff t = \pm \frac{2\sqrt{3}}{3} \]

Only \( t = \frac{2\sqrt{3}}{3} \) is in the interval \([0, 2]\), so using test points \( t = 0 \) and \( t = 2 \) we have

\[ x'(0) = -4 \leq 0 \implies x(t) \text{ is decreasing on } [0, \frac{2\sqrt{3}}{3}] \]
\[ x'(2) = 8 \geq 0 \implies x(t) \text{ is increasing on } [\frac{2\sqrt{3}}{3}, 2] \]

Since \( y(t) = t^2 \) then \( y(t) \geq 0 \) on \([0, 2]\).

(b) Give an expression for the area enclosed by the curve and the \( y \)-axis (but do not integrate).

\([0, \frac{2\sqrt{3}}{3}]\): \( x(t) \) is decreasing and \( y(t) \geq 0 \)
\([\frac{2\sqrt{3}}{3}, 2]\): \( x(t) \) is increasing and \( y(t) \geq 0 \)

Then the area enclosed by the curve and the \( y \)-axis is

\[ \int_{0}^{\frac{2\sqrt{3}}{3}} t^2(3t^2 - 4) \, dt + \int_{\frac{2\sqrt{3}}{3}}^{2} t^2(3t^2 - 4) \, dt = -\int_{0}^{\frac{2\sqrt{3}}{3}} t^2(3t^2 - 4) \, dt + \int_{\frac{2\sqrt{3}}{3}}^{2} t^2(3t^2 - 4) \, dt \]