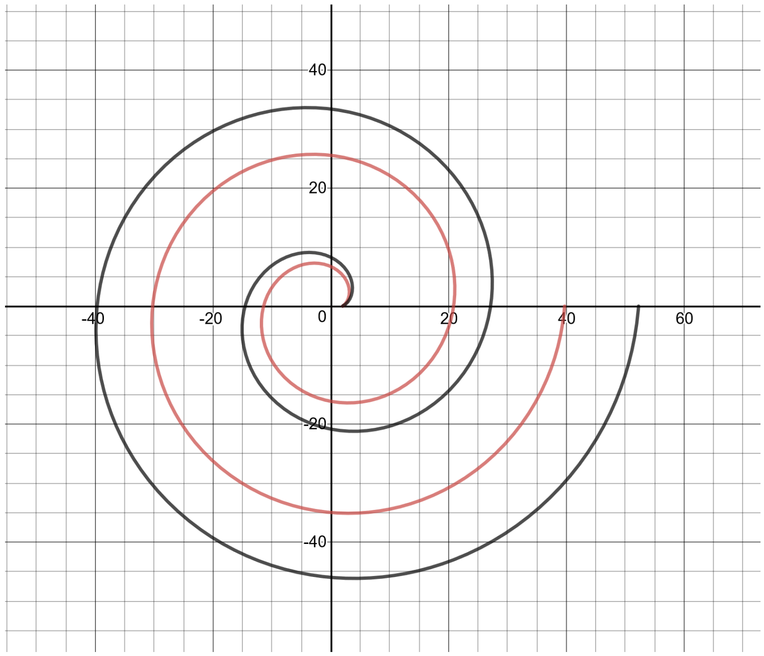
Friday Week 2

Calculus III



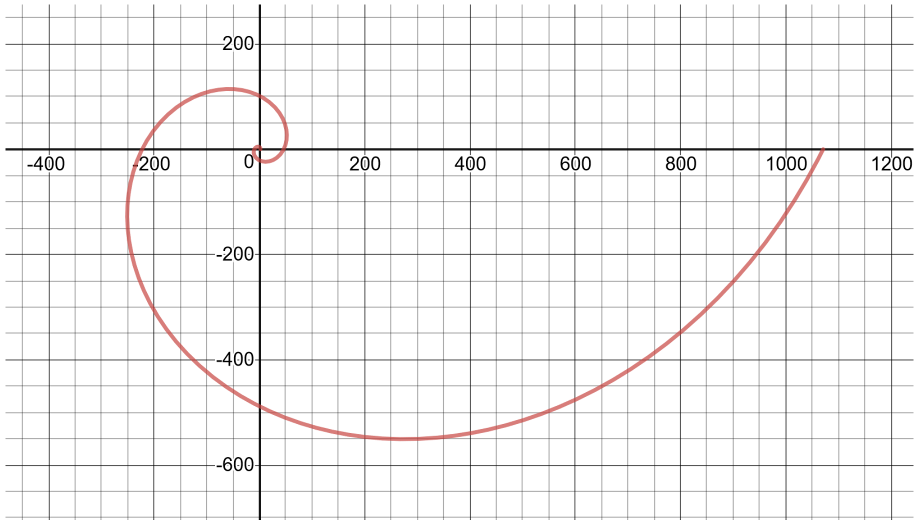
Above is a graph of two Archimedean spirals

$$r = 2 + 4\theta$$
$$r = 2 + 3\theta$$
$$\theta \in [0, 4\pi]$$

Find the area of the region between the two spirals on $[0, 2\pi]$. Shade this area on the graph.

Solution. The area is given by

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} \left((2+4\theta)^2 - (2+3\theta)^2\right) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(4 + 16\theta + 16\theta^2 - 4 - 12\theta - 9\theta^2\right) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(7\theta^2 + 4\theta\right) d\theta \\
&= \frac{1}{2}\left[\frac{7\theta^3}{3} + 2\theta^2\right]_0^{2\pi} \\
&= \frac{1}{2}\left(\frac{7(8\pi^3)}{3} + 2(4\pi^2)\right) \\
&= \frac{28\pi^3}{3} + 4\pi^2
\end{aligned}$$



Above is a graph of the logarithmic spiral $r = 2e^{\theta/2}$ on $[0, 4\pi]$. Find the arc length of this spiral on $[0, 2\pi]$.

Solution. The arc length is given by

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{(2e^{\theta/2})^2 + (e^{\theta/2})^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4e^{\theta} + e^{\theta}}\, d\theta \\
&= \int_0^{2\pi} \sqrt{5e^{\theta}}\, d\theta \\
&= \sqrt{5} \int_0^{2\pi} e^{\theta/2}\, d\theta \\
&= \sqrt{5} \left[2e^{\theta/2}\right]_0^{2\pi} \\
&= 2\sqrt{5}(e^{\pi} - e^{0}) \\
&= 2\sqrt{5}(e^{\pi} - 1)
\end{aligned}$$