Friday Week 4

Calculus III

Let $\{a_n\}$ be the sequence defined by $a_n = \frac{3-n}{4+n}$ for all $n \geq 1$.

(a) Show that $\{a_n\}$ is monotonic by showing that the function $f(x) = \frac{3-x}{4+x}$ is monotonic on $(1, \infty)$.

Solution. We have that

$$\begin{aligned} f'(x) &= \frac{(4+x)(-1) - (3-x)(1)}{(4+x)^2} \\ &= \frac{-7}{(4+x)^2} \end{aligned}$$

which is negative for all $x \in (1, \infty)$. Therefore $f(x)$ is decreasing on $(1, \infty)$, and since $a_n = f(n)$ then $\{a_n\}$ is decreasing as well. $\square$

(b) Show that $\{a_n\}$ is monotonic by comparing a_n and a_{n+1} for any $n \geq 1$.

Solution. From (a) we already know that $\{a_n\}$ is decreasing, so we will show that $a_{n+1} < a_n$ for all $n \geq 1$. (We could also compute a few terms of the sequence and guess that the sequence is decreasing.)

$$\begin{aligned} a_{n+1} < a_n &\iff \frac{3-(n+1)}{4+(n+1)} < \frac{3-n}{4+n} \\ &\iff \frac{2-n}{5+n} < \frac{3-n}{4+n} \\ &\iff (2-n)(4+n) < (3-n)(5+n) \\ &\iff 8 - 2n - n^2 < 15 - 2n - n^2 \\ &\iff 8 < 15 \end{aligned}$$

Since $8 < 15$, it follows that $a_{n+1} < a_n$ for all $n \geq 1$. Therefore $\{a_n\}$ is decreasing. $\square$

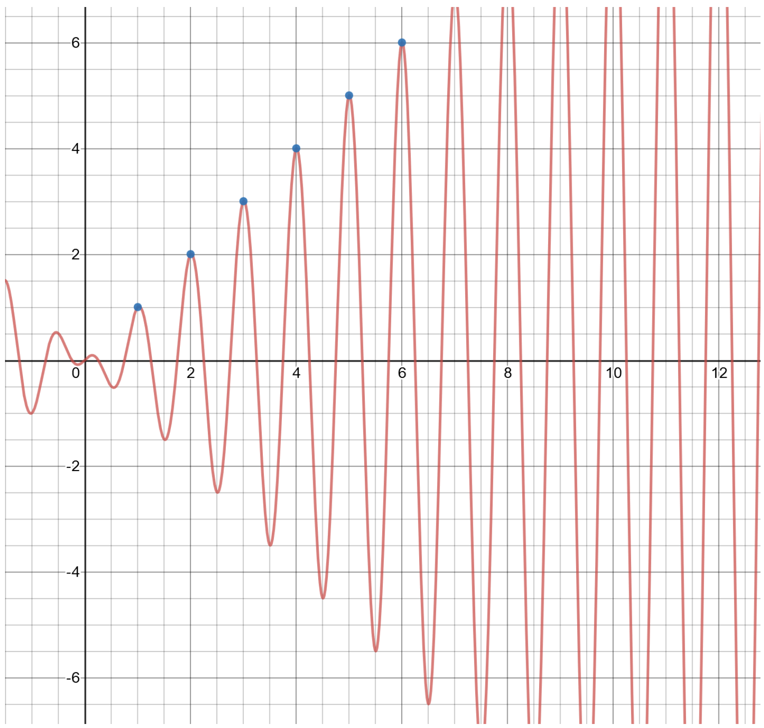
(c) Is $\{a_n\}$ is bounded above? bounded below? If so, find an upper bound and a lower bound.

Solution. Any decreasing sequence is bounded above by its first term, so $\{a_n\}$ is bounded above by $a_1 = \frac{3-1}{4+1} = \frac{2}{5}$. A lower bound is -1, the limit of the sequence. $\square$

For each statement below, indicate if the statement is true or false. If the statement is true, provide an explanation. If false, provide a counterexample.

1. **True or False.** If $a_n = f(n)$ for a function $f(x)$ and $\{a_n\}$ is monotonic, then $f(x)$ is monotonic on $(1, \infty)$.

   Solution. False. If $f(x)$ is monotonic, then so is $\{a_n\}$, but the converse is not true. For example, let $f(x) = x\sin(2\pi x + \frac{\pi}{2})$. If we define $\{a_n\}$ by $a_n = f(n)$, then $\{a_n\} = \{1, 2, 3, \dots\}$ is monotone increasing but $f(x)$ is not. $\square$



2. **True or False.** If $\{b_n\}$ diverges and $0 \le b_n \le a_n$ for all $n \ge 1$, then $\{a_n\}$ diverges.

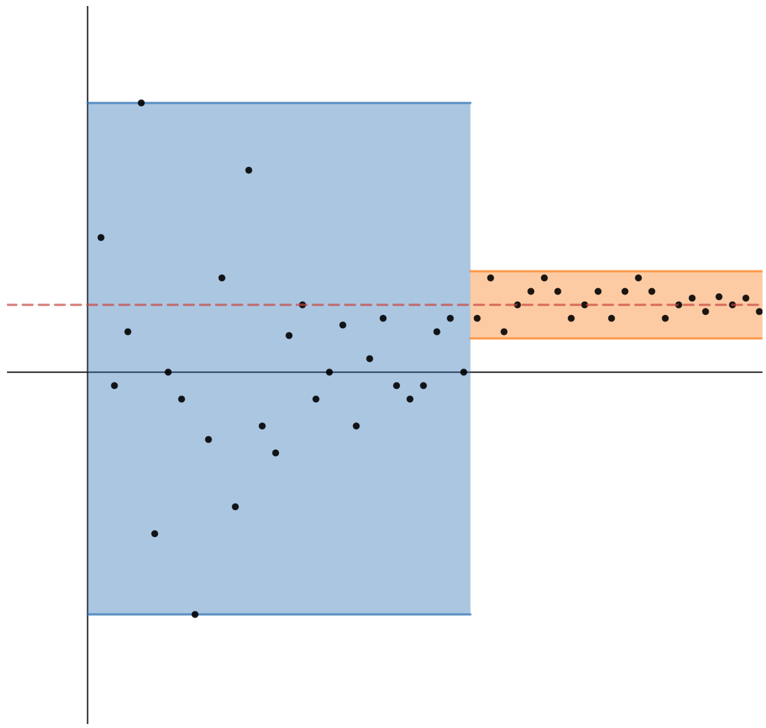
   Solution. False. If $\{b_n\}$ diverges to ∞ then $\{a_n\}$ also diverges to ∞, but it's possible that $\{b_n\}$ diverges without diverging to ∞. For example, take

$$\{b_n\} = \{0, 1, 0, 1, \dots\}$$
$$\{a_n\} = \{1, 1, 1, 1, \dots\}$$

   Then $0 \le b_n \le a_n$ for all $n \ge 1$ and $\{b_n\}$ diverges, but $\{a_n\}$ converges to 1. $\square$

3. **True or False.** Every unbounded sequence diverges.

Solution. True. This is equivalent to saying that every convergent sequence is bounded. If $\{a_n\}$ converges to a limit L, then eventually all of the terms in the sequence get close to L. Then there exists some $N \in \mathbb{N}$ such that $a_N, a_{N+1}, a_{N+2}, \ldots$ are all bounded (since they are close to L), and the terms $a_1, a_2, \ldots, a_{N-1}$ are also bounded (since they are bounded above by their largest term and bounded below by their smallest term). Therefore the entire sequence is bounded. $\square$



4. **True or False.** If $\{a_n\}$ is decreasing, $\{b_n\}$ is increasing, and $b_n \leq a_n$ for all $n \geq 1$, then $\{a_n\}$ and $\{b_n\}$ both converge.

Solution. True. This follows from the Monotonic Sequence Theorem. Both sequences are monotonic, and we can show that they are both bounded, hence they converge.

Since $\{a_n\}$ is decreasing then it is bounded above by a_1, i.e. $a_n \leq a_1$ for all $n \geq 1$. The term a_1 is also an upper bound for $\{b_n\}$ since $b_n \leq a_n$ for all $n \geq 1$. A lower bound of $\{b_n\}$ is its first term b_1 since $\{b_n\}$ is increasing, so together $\{b_n\}$ is a bounded sequence. Since $\{b_n\}$ is bounded and monotone, then $\{b_n\}$ converges.

We can similarly argue that $\{a_n\}$ converges. The sequence is bounded below by b_1 since $b_1 \leq b_n$ for all $n \geq 1$ and $b_n \leq a_n$ for all $n \geq 1$. So $\{a_n\}$ is bounded above by a_1 and bounded below by b_1. As a bounded and monotone sequence, $\{a_n\}$ converges. $\square$