Friday Week 8 - Solutions

Calculus III

1. $\displaystyle\sum_{n=0}^{\infty} \frac{n!(x+2)^n}{e^{2n}}$

   Suppose $x+2 \neq 0$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!(x+2)^{n+1}}{e^{2(n+1)}} \cdot \frac{e^{2n}}{n!(x+2)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)(x+2)}{e^2}\right| \\ &= \infty\end{aligned}$$

   By the Ratio Test, the series diverges (when $x+2 \neq 0$). If $x+2=0$, i.e. $x=-2$, then all terms of the series are 0, so the series converges to 0. Thus the power series converges when $x=-2$ and diverges for all other x-values.

   Center: -2
   Radius of convergence: 0
   Interval of convergence: $[-2,-2]=\{-2\}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n\sqrt{n+1}}$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{5^{n+1}\sqrt{(n+1)+1}} \cdot \frac{5^n\sqrt{n+1}}{(x-2)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x-2}{5}\cdot\frac{\sqrt{n+1}}{\sqrt{n+2}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x-2}{5}\cdot\sqrt{\frac{n+1}{n+2}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x-2}{5}\cdot\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}\right| \\ &= \left|\frac{x-2}{5}\cdot\sqrt{\frac{1+0}{1+0}}\right| \\ &= \left|\frac{x-2}{5}\right| \\ &= \frac{|x-2|}{5}\end{aligned}$$

   By the Ratio Test, the series converges if

$$\begin{aligned}\frac{|x-2|}{5} < 1 &\iff |x-2| < 5 \\ &\iff -5 < x-2 < 5 \\ &\iff -3 < x < 7\end{aligned}$$

and diverges if

$$\begin{aligned}\frac{|x-2|}{5} > 1 &\iff |x-2| > 5 \\ &\iff -(x-2) > 5 \text{ or } x-2 < 5 \\ &\iff x < -3 \text{ or } x > 7\end{aligned}$$

Hence the power series converges on $(-3, 7)$. Note that we found that the series converges if $|x-2| < 5$, which is in the form $|x-a| < R$, where a is the center and R is the radius of convergence. So we can read off that the series is centered at 2 and the radius of convergence is 5. Finally we need to see if the series converges or diverges at each endpoint.

Left endpoint $x = -3$:

The series is $\sum_{n=0}^{\infty} \frac{(-3-2)^n}{5^n\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-5)^n}{5^n\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, so we may try the Alternating Series Test. We have that $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ as required, and $\frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{(n+1)+1}}$ for all $n \geq 0$. Therefore the series is convergent. This shows that the power series converges on $[-3, 7)$, and we are left to check for convergence at the right endpoint.

Right endpoint $x = 7$:

The series is $\sum_{n=0}^{\infty} \frac{(7-2)^n}{5^n\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{5^n}{5^n\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series with $p = \frac{1}{2}$. So the power series does not converge when $x = 7$, thus the interval of convergence is $[-3, 7)$.

Center: 2
Radius of convergence: 5
Interval of convergence: $[-3, 7)$

3. $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-1)^n}\right| \\ &= \lim_{n\to\infty} \left|\frac{x-1}{n+1}\right| \\ &= 0\end{aligned}$$

Since the limit is 0 for all values of x, then by the Ratio Test the power series converges for all values of x. Hence the interval of convergence is $(-\infty, \infty) = \mathbb{R}$, centered at 1 with an infinite radius of convergence.

Note: We determine the center of the interval of convergence directly from the power series as it's presented. A power series is of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ where a is the center of the interval of convergence, so in this case we have $c_n = \frac{1}{n!}$, and center $a = 1$.

Center: 1
Radius of convergence: ∞
Interval of convergence: $(-\infty, \infty) = \mathbb{R}$