Homework 4 - Solutions

Calculus III

Exercise 1

Solution.

$$\begin{aligned}\int_e^\infty \frac{1}{x(\ln x)^p}\,dx &= \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^p}\,dx\\ &= \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u^p}\,du\end{aligned}$$

If $p = 1$, then the integral is

$$\begin{aligned}\lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u}\,du &= \lim_{t\to\infty}\left[\ln u\right]_1^{\ln t}\\ &= \lim_{t\to\infty}\left(\ln(\ln t) - \ln(1)\right)\\ &= \infty\end{aligned}$$

hence the integral diverges. If $p \neq 1$, then the integral is

$$\begin{aligned}\lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u^p}\,du &= \lim_{t\to\infty}\left[\frac{u^{-p+1}}{-p+1}\right]_1^{\ln t}\\ &= \lim_{t\to\infty}\left(\frac{(\ln t)^{-p+1}}{-p+1} - \frac{1}{-p+1}\right)\end{aligned}$$

Here we consider two cases. If $-p+1 > 0$, then

$$\begin{aligned}\lim_{t\to\infty}\left(\frac{(\ln t)^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) &= \infty - \frac{1}{-p+1}\\ &= \infty\end{aligned}$$

hence the integral diverges. If $-p+1 < 0$, so $p-1 > 0$, then

$$\begin{aligned}\lim_{t\to\infty}\left(\frac{(\ln t)^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) &= \lim_{t\to\infty}\left(\frac{1}{(-p+1)(\ln t)^{p-1}} - \frac{1}{-p+1}\right)\\ &= 0 - \frac{1}{-p+1}\\ &= \frac{1}{p-1}\end{aligned}$$

hence the integral converges. Therefore the integral diverges when $p \leq 1$ and converges to $\frac{1}{p-1}$ when $p > 1$. □

Exercise 2

Solution.

$$\begin{aligned}
\int_0^\infty \frac{1}{x^3+\sqrt{x}}\,dx &= \int_0^1 \frac{1}{x^3+\sqrt{x}}\,dx + \int_1^\infty \frac{1}{x^3+\sqrt{x}}\,dx \\
&\le \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{x^3}\,dx \\
&= \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx + \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx \\
&= \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^1 + \lim_{t\to\infty} \left[\frac{1}{-2x^2}\right]_1^t \\
&= \lim_{t\to 0^+} \left(2-2\sqrt{t}\right) + \lim_{t\to\infty} \left(\frac{1}{-2t^2}+\frac{1}{2}\right) \\
&= 2+\frac{1}{2} \\
&= \frac{5}{2}
\end{aligned}$$

Since $0 \le \frac{1}{x^3+\sqrt{x}}$ on $(0,\infty)$, $\int_0^\infty \frac{1}{x^3+\sqrt{x}}\,dx \le \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{x^3}\,dx$, and both $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ and $\int_1^\infty \frac{1}{x^3}\,dx$ converge, then $\int_0^\infty \frac{1}{x^3+\sqrt{x}}\,dx$ converges by the Comparison Test. □

Exercise 3

Solution.

(a) The first five terms are

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= 1+\frac{1}{1} \\
x_3 &= 1+\frac{1}{1+\frac{1}{1}} \\
x_4 &= 1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}} \\
x_5 &= 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}
\end{aligned}$$

(b) The first five terms as single fractions are

$$\begin{aligned}
x_1 &= 1\\
x_2 &= 2\\
x_3 &= \frac{3}{2}\\
x_4 &= \frac{5}{3}\\
x_5 &= \frac{8}{5}
\end{aligned}$$

Rewriting as a list $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}$ we might see that the denominators are forming the Fibonacci sequence $(F_n) = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots)$, and the numerators are also forming the Fibonacci sequence but beginning with its second term.

(c)

$$\begin{aligned}
x_1 &= 1\\
x_2 &= 2\\
x_3 &= 1.5\\
x_4 &= 1.\overline{6}\\
x_5 &= 1.6\\
x_6 &= \frac{13}{8} = 1.625\\
x_7 &= \frac{21}{13} \approx 1.61538461538\\
x_8 &= \frac{34}{21} \approx 1.61904761905\\
x_9 &= \frac{55}{34} \approx 1.61764705882
\end{aligned}$$

The sequence seems to be converging. Indeed, the sequence converges to the golden ratio $\varphi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$. □