Homework 5 - Solutions

Calculus III

Exercise 1

Solution. We have that

$$\begin{aligned}\frac{2^n}{n!} &= \frac{2\cdot 2\cdot 2\cdots 2\cdot 2}{n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1}\\ &= \frac{2}{n}\cdot\frac{2}{n-1}\cdot\frac{2}{n-2}\cdots\frac{2}{2}\cdot\frac{2}{1}\\ &\le \frac{2}{n}\cdot 1\cdot 1\cdots 1\cdot\frac{2}{1}\\ &= \frac{4}{n}\end{aligned}$$

where the above inequality holds since $\frac{2}{n-1}\le 1$, $\frac{2}{n-2}\le 1$, $\ldots$, $\frac{2}{3}\le 1$, and $\frac{2}{2}\le 1$. Then

$$0 \le \frac{n!}{2^n} \le \frac{4}{n}$$

where $\lim_{n\to\infty} 0 = \lim_{n\to\infty}\frac{4}{n} = 0$. Therefore $\lim_{n\to\infty}\frac{2^n}{n!} = 0$ by the Squeeze Theorem.

Exercise 2

Solution. Step 1: remove 1 square of area $\left(\frac{1}{3}\right)^2 \implies$ area of $\frac{1}{9}$ removed
Step 2: remove 8 squares of area $\left(\frac{1}{3^2}\right)^2 \implies$ area of $\frac{8^1}{9^2}$ removed
Step 3: remove 8^2 squares of area $\left(\frac{1}{3^3}\right)^2 \implies$ area of $\frac{8^2}{9^3}$ removed
Step n: remove 8^{n-1} squares of area $\left(\frac{1}{3^n}\right)^2 \implies$ area of $\frac{8^{n-1}}{3^{2n}} = \frac{8^{n-1}}{9^n}$ removed

The total area removed is found by summing the areas removed at each step:

$$\sum_{n=1}^{\infty}\frac{8^{n-1}}{3^{2n}} = \sum_{n=1}^{\infty}\frac{8^{n-1}}{9^n} = \sum_{n=1}^{\infty}\frac{1}{9}\cdot\left(\frac{8}{9}\right)^{n-1}$$

This is a geometric series with $a=\frac{1}{9}$ and $r=\frac{8}{9}<1$ that converges to $\dfrac{a}{1-r} = \dfrac{\frac{1}{9}}{1-\frac{8}{9}} = 1$. The original area was 1 and the total area removed is 1, so the total area of the Sierpinski Carpet is $1-1=0$.