Exercise 1

Solution. We have that

\[
\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2}{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}
\]

\[
= \frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdots \frac{2}{1}
\]

\[
\leq \frac{2}{n} \cdot \frac{1}{1} \cdots \frac{2}{1}
\]

\[
= \frac{4}{n}
\]

where the above inequality holds since \( \frac{2}{n-1} \leq 1, \frac{2}{n-2} \leq 1, \ldots, \frac{2}{3} \leq 1, \) and \( \frac{2}{2} \leq 1. \) Then

\[
0 \leq \frac{n!}{2^n} \leq \frac{4}{n}
\]

where \( \lim_{n \to \infty} 0 = \lim_{n \to \infty} \frac{4}{n} = 0. \) Therefore \( \lim_{n \to \infty} \frac{2^n}{n!} = 0 \) by the Squeeze Theorem.

Exercise 2

Solution. Step 1: remove 1 square of area \( \left( \frac{1}{3} \right)^2 \) \( \implies \) area of \( \frac{1}{9} \) removed

Step 2: remove 8 squares of area \( \left( \frac{1}{3^2} \right)^2 \) \( \implies \) area of \( \frac{8}{9^2} \) removed

Step 3: remove \( 8^2 \) squares of area \( \left( \frac{1}{3^3} \right)^2 \) \( \implies \) area of \( \frac{8^2}{9^3} \) removed

Step n: remove \( 8^{n-1} \) squares of area \( \left( \frac{1}{3^n} \right)^2 \) \( \implies \) area of \( \frac{8^{n-1}}{9^{2n}} = \frac{8^{n-1}}{9^n} \) removed

The total area removed is found by summing the areas removed at each step:

\[
\sum_{n=1}^{\infty} \frac{8^{n-1}}{3^{2n}} = \sum_{n=1}^{\infty} \frac{8^{n-1}}{9^n} = \sum_{n=1}^{\infty} \frac{1}{9} \cdot \left( \frac{8}{9} \right)^{n-1}
\]

This is a geometric series with \( a = \frac{1}{9} \) and \( r = \frac{8}{9} < 1 \) that converges to \( \frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{8}{9}} = 1. \)

The original area was 1 and the total area removed is 1, so the total area of the Sierpinski Carpet is \( 1 - 1 = 0. \)