#1 Solution. In the graph below, the area of the rectangles is \(\frac{1}{3\ln3} + \frac{1}{4\ln4} + \frac{1}{5\ln5} + \cdots + \frac{1}{N\ln N}\). This area is an overestimate of \(\int_3^{N+1} \frac{1}{x\ln x} \, dx\) since \(y = \frac{1}{x\ln x}\) is decreasing on \((3, N + 1)\).

Since we know that \(\frac{1}{3\ln3} + \frac{1}{4\ln4} + \frac{1}{5\ln5} + \cdots + \frac{1}{N\ln N} > \int_3^{N+1} \frac{1}{x\ln x} \, dx\), then to find a value of \(N\) so that \(\frac{1}{3\ln3} + \frac{1}{4\ln4} + \frac{1}{5\ln5} + \cdots + \frac{1}{N\ln N} > 3\), it suffices to find the smallest value of \(N\) such that \(\int_3^{N+1} \frac{1}{x\ln x} \, dx > 3\). Evaluating the integral, we have that

\[
\int_3^{N+1} \frac{1}{x\ln x} \, dx = \int_{\ln(3)}^{\ln(N+1)} \frac{1}{u} \, du = \ln(u) \bigg|_{\ln(3)}^{\ln(N+1)} = \ln(N+1) - \ln(3).
\]

Then

\[
\ln(N+1) - \ln(3) > 3 \iff \ln(N+1) > \ln(3) + 3 \iff e^{\ln(N+1)} > e^{\ln(3)+3} \iff \ln(N+1) > \ln(3)e^3 \iff e^{\ln(N+1)} > e^{\ln(3)e^3} \iff N + 1 > 3e^3 \iff N > 3e^3 - 1 \approx 3830333408.01.
\]

Therefore \(N = 3830333409\) guarantees that \(\frac{1}{3\ln3} + \frac{1}{4\ln4} + \cdots + \frac{1}{N\ln N} > 3\).

#2 Solution. Let \(f(x) = \frac{1}{x^4}\). Then \(f(x)\) is positive on \((1, \infty)\), and \(f(x)\) is decreasing on \((1, \infty)\) since \(f'(x) = -\frac{4}{x^5} < 0\). Since the conditions for the Integral Test are satisfied, we may use the Integral Test approximation formula. To keep the error within 0.001, we set
\[
\int_N^{\infty} \frac{1}{x^4} \, dx < 0.001.
\]
We have that
\[
\int_N^{\infty} \frac{1}{x^4} \, dx = \lim_{t \to \infty} \int_N^t \frac{1}{x^4} \, dx = \lim_{t \to \infty} \left. \frac{1}{-3x^3} \right|_N^t = \lim_{t \to \infty} \left( \frac{1}{-3t^3} + \frac{1}{3N^3} \right) = \frac{1}{3N^3}.
\]
Then
\[
\int_N^{\infty} \frac{1}{x^4} \, dx < 0.001 \iff \frac{1}{3N^3} < 0.001
\]
\[\iff \frac{1}{3N^3} < \frac{1}{1000}\]
\[\iff 1000 < 3N^3\]
\[\iff \frac{1000}{3} < N^3\]
\[\iff N > \left( \frac{1000}{3} \right)^{\frac{1}{3}} \approx 6.9\]

Then take \(N = 7\). The 7th partial sum estimation is
\[
\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} \approx 1.08154.
\]

We see that this is within 0.001 of the actual sum of the series, \(\pi^4 \approx \frac{90}{\pi^2} \approx 1.08232\), as expected.

\#3 Solution. We wish to show that \(\sum_{n=2}^{\infty} \frac{1}{n^2} \) converges by the Comparison Test. We have that
\[
0 < \frac{1}{n!} = \frac{1}{n \cdot (n - 1) \cdot \ldots \cdot 2 \cdot 1} = \frac{1}{n} \cdot \frac{1}{n - 1} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{1}{1} \leq \frac{1}{n} \cdot \frac{1}{n - 1} = \frac{1}{n^2 - n}
\]
Hence if \(\sum_{n=2}^{\infty} \frac{1}{n^2 - n}\) converges, then \(\sum_{n=2}^{\infty} \frac{1}{n!}\) converges by the Comparison Test. We will show that \(\sum_{n=2}^{\infty} \frac{1}{n^{2} - n}\) converges by the Limit Comparison Test. Let \(a_n = \frac{1}{n^{2} - n}\) and \(b_n = \frac{1}{n^2}\). Then \(a_n, b_n > 0\), and the limit
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{n^{2} - n}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2}{n^2 - n} = \lim_{n \to \infty} \frac{1}{1 - \frac{1}{n}} = 1
\]
is positive and finite. Since \(\sum_{n=2}^{\infty} \frac{1}{n^2}\) converges as a \(p\)-series with \(p = 2\), then \(\sum_{n=2}^{\infty} \frac{1}{n^{2} - n}\) converges by the Limit Comparison Test. Therefore \(\sum_{n=2}^{\infty} \frac{1}{n!}\) converges.