Homework 7 - Solutions

Calculus III

#1 Solution.

(a) The series is convergent by the Alternating Series Test since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n} > \frac{1}{n+1}$ for all $n \geq 1$.

(b) To estimate the series within 0.1 of its true value, we want to find an N such that $x_{N+1} < 0.1$. We have that

$$\begin{aligned} x_{N+1} < 0.1 &\iff \frac{1}{N+1} < 0.1 \\ &\iff \frac{1}{N+1} < \frac{1}{10} \\ &\iff 10 < N+1 \\ &\iff 9 < N \end{aligned}$$

so we may take $N = 10$. The estimation of the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \approx 0.64563$.

(c) The series converges to $\ln 2 \approx 0.69315$, which is within 0.1 of the estimated value.

#2 Solution. The "positive part" of the alternating harmonic series is

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

This is a divergent series. One way to show divergence is by the Comparison Test: $0 < \frac{1}{2n} < \frac{1}{2n-1}$ for all $n \geq 1$, and the series $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. It follows by the Comparison Test that $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges.

Another way to show divergence is by the Limit Comparison Test. Let $a_n = \frac{1}{2n-1}$ and $b_n = \frac{1}{n}$, noting that $a_n > 0$ and $b_n > 0$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n}{2n-1} \\ &= \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} \\ &= \frac{1}{2} \end{aligned}$$

This limit is positive and finite, so by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ both converge or both diverge. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges.

A third way to show divergence is by the Integral Test, letting $f(x) = \frac{1}{2x-1}$. We could check that $f(x)$ is positive and decreasing on $(1, \infty)$ and then check that $\int_1^{\infty} \frac{1}{2x-1}\, dx$ diverges.

The "negative part" of the alternating harmonic series is

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots=\sum_{n=1}^{\infty}\left(-\frac{1}{2n}\right)$$

This series diverges since $\sum_{n=1}^{\infty}\left(-\frac{1}{2n}\right)=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

#3 Solution. We will determine whether the series $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ converges or diverges by the Ratio Test. Letting $a_n=\frac{(n!)^2}{(2n)!}$, we have that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} \\
&= \lim_{n\to\infty}\frac{((n+1)!)^2(2n)!}{(2(n+1))!(n!)^2} \\
&= \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\right)^2\frac{(2n)!}{(2n+2)!} \\
&= \lim_{n\to\infty}(n+1)^2\cdot\frac{1}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{2(n+1)(2n+1)} \\
&= \lim_{n\to\infty}\frac{n+1}{2(2n+1)} \\
&= \lim_{n\to\infty}\frac{n+1}{4n+2} \\
&= \lim_{n\to\infty}\frac{1+\frac{1}{n}}{4+\frac{2}{n}} \\
&= \frac{1}{4} \\
&< 1
\end{aligned}$$

By the Ratio Test, the series converges (absolutely).