MATH 1953 Written Homework 8 Solutions

1. Suppose a student is able to correctly show that a series $\sum_{n=1}^{\infty} a_n$ converges using the Root Test. Explain why that student should be able to conclude that the series $\sum_{n=1}^{\infty} na_n$ converges as well.

Solution: If the Root Test shows that the series $\sum_{n=1}^{\infty} a_n$ converges, this means that $\sqrt[n]{|a_n|}$ converges to a limit $L < 1$. Then, when we apply the Root Test to $\sum_{n=1}^{\infty} na_n$, we get

$$\sqrt[n]{|na_n|} = \sqrt[n]{n}\,\sqrt[n]{|a_n|}.$$

Since $\sqrt[n]{n} \to 1$, the limit of this will be the same limit L as we got for only $\sqrt[n]{|a_n|}$. We know that this limit L is less than 1, and so the Root Test shows that the series $\sum_{n=1}^{\infty} na_n$ converges as well.

2. For each of the series from problems #11-19 in the Chapter 11 Review Problems in your textbook (the first one should be the sum of $\frac{n}{n^3+1}$), indicate which convergence test (or tests; some may require more than one!) you would use to determine whether the series converges or diverges. If using Comparison/Limit Comparison Test, indicate what series you would compare to.

You DO NOT need to show the work of applying the test to conclude convergence/divergence. However, you should convince yourself on scratch paper that the test you want would actually yield an answer, as you'll only receive credit for tests that could successfully be applied.

Solution:

- 11: Comparison or Limit Comparison Test with $y_n = \frac{1}{n^2}$
- 12: Limit Comparison Test with $y_n = \frac{1}{n}$
- 13: Root or Ratio Test
- 14: Alternating Series Test
- 15: Integral Test
- 16: Divergence Test
- 17: Absolute Convergence Test, followed by Comparison or Limit Comparison Test with $y_n = \left(\frac{1}{1.2}\right)^n$
- 18: Root Test
- 19: Ratio Test

3. What is the radius of convergence R of the power series $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$, i.e. the value of R so that the series converges for all x with $|x| < R$ and diverges for all x with $|x| > R$?

Solution: We use the Ratio Test:

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\frac{(n+1)^{n+1}|x|^{n+1}}{(n+1)!}}{\frac{n^n|x|^n}{n!}} = \frac{(n+1)^{n+1}|x|^{n+1}n!}{(n+1)!n^n|x|^n} = \frac{(n+1)^{n+1}|x|}{(n+1)n^n} = \frac{(n+1)^n|x|}{n^n} = \left(1+\frac{1}{n}\right)^n |x|.$$

Now, we need the limit of $\left(1+\frac{1}{n}\right)^n$, for which we use L'Hospital's Rule. First we convert to x and take an ln to get $x \ln\left(1+\frac{1}{x}\right)$. Then, we rewrite as a fraction so we can use L'Hospital's Rule:

$$\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1+1/x}(-x^{-2})}{-x^{-2}} = \lim_{x\to\infty} \frac{1}{1+1/x} = 1.$$

Remember that we need to "undo the ln," so the original limit is actually e^1 or e.

Since $\left(1+\frac{1}{n}\right)^n \to e$, we see that the limit from the Ratio Test is $e|x|$. For convergence, we set this less than 1:

$$e|x| < 1 \longleftrightarrow |x| < \frac{1}{e}.$$

So, the radius of convergence of this series is $\frac{1}{e}$ (and the center is 0).