MATH 3162 Homework Assignment 1 Solutions

4.4.1(a): Choose any $x_0 \in \mathbb{R}$; we wish to show that $f(x) = x^3$ is continuous at x_0. Take an arbitrary sequence (x_n) approaching x_0. Then, since we can multiply convergent sequences, we know that $(x_n^3) \to x_0^3$, i.e. $f(x_n) \to x_0$. This proves continuity of f at x_0, and since x_0 was arbitrary, on all of $\mathbb{R}$.

ALTERNATE PROOF: You can also do this with ϵ and δ. Choose any $\epsilon > 0$. Define $\delta = \min(1, \frac{\epsilon}{3|x_0|^2+3|x_0|+1})$. Consider any $x \in \mathbb{R}$ with $|x - x_0| < \delta$. Then, clearly $|x - x_0| < 1$ and $|x - x_0| < \frac{\epsilon}{3|x_0|^2+3|x_0|+1}$. Since $|x - x_0| < 1$, $|x| \leq |x_0| + |x - x_0| \leq |x_0| + 1$. Then,

$$|x^3 - x_0^3| = |x - x_0| \cdot |x^2 + x_0 x + x_0^2| \leq \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1}(|x|^2 + |x_0||x| + |x_0|^2)$$

$$\leq \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1}((|x_0|+1)^2 + |x_0|(|x_0|+1) + |x_0|^2) \leq \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1}(3|x_0|^2 + 3|x_0| + 1) = \epsilon.$$

This completes the proof that f is continuous at x_0, and since x_0 was arbitrary, we've shown that f is continuous on $\mathbb{R}$.

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4.4.1(b): Define $\epsilon = 1$. We will exhibit sequences x_n and y_n so that $|x_n - y_n| \to 0$, but for all n, $|f(x_n) - f(y_n)| \geq 1$. For every $n \in \mathbb{N}$, define $y_n = n$ and $x_n = n + \frac{1}{3n^2}$. Then clearly $|x_n - y_n| = \frac{1}{3n^2}$, which indeed approaches 0. Also, for every $n \in \mathbb{N}$,

$$|f(x_n) - f(y_n)| = \left| \left(n + \frac{1}{3n^2} \right)^3 - n^3 \right| = \left| n^3 + 1 + \frac{1}{3n^3} + \frac{1}{27n^6} - n^3 \right| = 1 + \frac{1}{3n^3} + \frac{1}{27n^6} \geq 1.$$

By Theorem 4.4.5, this means that f is not uniformly continuous on $\mathbb{R}$.

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4.4.1(c): Suppose that S is a bounded subset of $\mathbb{R}$, meaning that there exists N so that $S \subseteq [-N, N]$. Since f is continuous on $[-N, N]$ and $[-N, N]$ is compact, f is uniformly continuous on $[-N, N]$. However, this clearly means that f is uniformly continuous on S as well; since f is u.c. on $[-N, N]$, $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. $(|x - y| < \delta$ and $x, y \in [-N, N]) \Longrightarrow |f(x) - f(y)| < \epsilon$, and this clearly implies the same statement for S since $S \subseteq [-N, N]$.

ALTERNATE PROOF: if you want to prove this directly without using Theorem 4.4.7, it's also possible. Suppose that $S \subseteq [-N, N]$. Choose any $\epsilon > 0$, and define $\delta = \frac{\epsilon}{3N^2}$. Then, assume that $x, y \in S$ and $|x - y| < \delta$. Since $x, y \in S$, $|x|, |y| \leq N$. Then,

$$|x^3 - y^3| = |x - y| \cdot |x^2 + xy + y^2| \leq \frac{\epsilon}{3N^2}(|x|^2 + |x||y| + |y|^2) \leq \frac{\epsilon}{3N^2} \cdot 3N^2 = \epsilon.$$

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4.4.5: Assume that $g : (a, c) \to \mathbb{R}$ is uniformly continuous on $(a, b]$ and $[b, c)$. Choose any $\epsilon > 0$. Since g is u.c. on $(a, b]$, there exists δ_1 so that for all $x, y \in (a, b]$ where $|x - y| < \delta_1$, it is true that $|g(x) - g(y)| < \frac{\epsilon}{2}$. Similarly, since g is u.c. on $[b, c)$, there exists δ_2 so that for all $x, y \in [b, c)$ where $|x - y| < \delta_2$, it is true that $|g(x) - g(y)| < \frac{\epsilon}{2}$. Now, define $\delta = \min(\delta_1, \delta_2)$. Assume that $x, y \in (a, c)$ and $|x - y| < \delta$; clearly then $|x - y| < \delta_1$ and $|x - y| < \delta_2$. There are four cases.

Case 1: $x, y \in (a, b]$. Then, $x, y \in (a, b]$ and $|x - y| < \delta_1$, meaning that $|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon$.

Case 2: $x, y \in [b, c)$. Then, $x, y \in [b, c)$ and $|x - y| < \delta_2$, meaning that $|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon$.

Case 3: $x \in (a, b]$ and $y \in [b, c)$. Then, $x, b \in (a, b]$ and $|x - b| \leq |x - y| < \delta_1$, meaning that $|g(x) - g(b)| < \frac{\epsilon}{2}$. Similarly, $b, y \in [b, c)$ and $|b - y| \leq |x - y| < \delta_2$, meaning that $|g(b) - g(y)| < \frac{\epsilon}{2}$. Finally,

$$|g(x)-g(y)| = |g(x)-g(b)+g(b)-g(y)| \leq |g(x)-g(b)|+|g(b)-g(y)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

Case 4: $y \in (a, b]$ and $x \in [b, c)$. Simply reverse the names of x and y and apply the proof from Case 3 to see that $|g(x) - g(y)| < \epsilon$.

We showed that for every $x, y \in (a, c)$ with $|x - y| < \delta$, $|g(x) - g(y)| < \epsilon$, so we've proved uniform continuity of g on (a, c).

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Extra problem 1: (Sequential definition) Define $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. The sequence $x_n = \frac{1}{n}$, i.e. $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is in S. However, it converges to 0, and so all of its subsequences converge to 0 as well. This means that (x_n) does not have a convergent subsequence converging to a limit in S, verifying that S is not compact.

(Open cover definition) For every n, define $U_n = \left(\frac{n+0.5}{n(n+1)}, \frac{n-0.5}{n(n-1)}\right)$. Note that for every n, $\frac{1}{n} \in U_n$. Therefore, $\bigcup_{n=1}^{\infty} U_n \supseteq \bigcup_{n=1}^{\infty} \{\frac{1}{n}\} = S$, i.e. the collection $U_1, U_2, U_3, \ldots$ forms an open cover for S. For every n, the right endpoint of U_{n+1} is $\frac{(n+1)-0.5}{(n+1)(n+1-1)} = \frac{n+0.5}{n(n+1)}$, which is the left endpoint of U_n, and so all of the sets U_n are disjoint from one another. Therefore, for every $m \neq n$, $\frac{1}{n} \notin U_m$. This means that if we remove any particular set U_n from the collection $U_1, U_2, \ldots$, the union of the remaining sets will not contain $\frac{1}{n}$ and therefore will not contain S. So, there is no finite subcover of the open cover $U_1, U_2, \ldots$, meaning that S is not compact.

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Extra problem 2: Assume that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are functions which are each uniformly continuous on D. Choose any $\epsilon > 0$. Since f is u.c. on D, there exists δ_1 so that for all $x, y \in D$ with $|x-y| < \delta_1$, it is the case that $|f(x) - f(y)| < \frac{\epsilon}{2}$. Similarly, since g is u.c. on D, there exists δ_2 so that for all $x, y \in D$ with $|x - y| < \delta_2$, it is the case that $|g(x) - g(y)| < \frac{\epsilon}{2}$. Now, define $\delta = \min(\delta_1, \delta_2)$. Consider any $x, y \in D$ for which $|x - y| < \delta$. Clearly then $|x-y| < \delta_1$ and $|x-y| < \delta_2$. Therefore, $|f(x)-f(y)| < \frac{\epsilon}{2}$ and $|g(x)-g(y)| < \frac{\epsilon}{2}$, meaning that

$$|(f+g)(x)-(f+g)(y)| = |f(x)+g(x)-f(y)-g(y)| = |(f(x)-f(y))+(g(x)-g(y))|$$
$$\leq |f(x) - f(y)| + |g(x) - g(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since ϵ was arbitrary, this verifies uniform continuity of $f + g$ on D.

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Extra problem 3: Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ have the property that $\exists \delta > 0$ s.t. $\forall \epsilon > 0$, $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. Consider any $x \in (x_0 - \delta, x_0 + \delta)$; clearly $|x - x_0| < \delta$. Suppose for a contradiction that $f(x) \neq f(x_0)$, and set $\epsilon := |f(x) - f(x_0)|$; clearly $\epsilon > 0$. Since $\epsilon > 0$ and $|x - x_0| < \delta$, by assumption we know that $|f(x) - f(x_0)| < \epsilon$. However, this means that $|f(x) - f(x_0)| < |f(x) - f(x_0)|$, a contradiction. Therefore, our assumption was wrong and $f(x) = f(x_0)$. Since $x \in (x_0 - \delta, x_0 + \delta)$ was arbitrary, we've shown that $f(x) = f(x_0)$ for every such x, i.e that f is constant on $(x_0 - \delta, x_0 + \delta)$.

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