Math 3162 Homework Assignment 2 Extra Problem Solutions

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1 4.4.13(b) Problem Statement

Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$.

1.1 Notes

In the proof I claim that if (x_n) is not Cauchy, then there exists an $\epsilon > 0$ and a subsequence (x_{n_k}) such that for all k, $|(x_{n_k}) - (x_{n_{k+1}})| > \epsilon$. If you aren't sure you can prove this, you should read the proof directly below:
Proof: If (x_n) is Cauchy then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m > \mathbb{N}$, $|(x_{n_k}) - (x_{n_{k+1}})| < \epsilon$.

If (x_n) is NOT Cauchy then there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists an $n, k > N$ such that $|(x_n) - (x_k)| \geq \epsilon$. By the triangle inequality, either $|(x_N) - (x_k)| \geq \frac{\epsilon}{2}$ or $|(x_N) - (x_n)| \geq \frac{\epsilon}{2}$. Now we can say that if (x_n) is NOT Cauchy then there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists a $k > N$ such that $|(x_N) - (x_k)| \geq \epsilon$. Now we can define a subsequence (a_{n_k}) by choosing $n_1 = 1$ and defining n_{k+1} to be the smallest n value greater than n_k, such that $|a_{n_{k+1}} - a_{n_k}| > \epsilon$.

1.2 Solution

$\Leftarrow$ Direct Proof

Suppose there exist $g(a)$ and $g(b)$ such that g is continuous on $[a, b]$. Since g is continuous, and $[a, b]$ is compact, we know that g is uniformly continuous on $[a, b]$. Since $(a, b) \subset [a, b]$, by the definition of uniform continuity, we know that g is also uniformly continuous on (a, b).
$\Rightarrow$ Proof By Contraposition
Suppose there does not exist $g(a)$ and $g(b)$ so that g is continuous on $[a, b]$. Without loss of generality, suppose that an appropriate $g(a)$ does not exist. Then it must be because $lim_{x \to a^+} g(x)$ does not exist. That means there exists an $a_n \to a$ such that $g(a_n)$ does not converge. Therefore, $g(a_n)$ is not Cauchy. Then there exists a $\epsilon > 0$ and a subsequence $(a_{n_k}) \to a$ such that for all $k \in \mathbb{N}$, $|g(a_{n_k}) - g(a_{n_{k+1}})| > \epsilon$. Define $a'_{n_k} = a_{n_{k+1}}$. Since (a_n) converges, (a_{n_k}) is Cauchy. So $|a_{n_k} - a'_{n_k}| \to 0$, but $|g(a_{n_k}) - g(a'_{n_k})| > \epsilon$. By the sequential criterion for absence of uniform continuity, g is not uniformly continuous on (a, b).

2 4.5.8 Problem Statement

If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$. Show that if f is continuous on an interval $[a, b]$ and one-to-one, then f^{-1} is continuous.

2.1 Notes

We can actually show something even stronger: If f is continuous on an compact set, A, and one-to-one, then f^{-1} is continuous.

2.2 Solution

By way of contradiction, suppose that $f^{-1} : f(A) \to A$ is not continuous. Then there exists an $x \in A$ and sequence $(y_n) \subseteq f(A)$ such that $(y_n) \to f(x)$ but $f^{-1}(y_n) \not\to f^{-1}(f(x))$. Then $f^{-1}(y_n) \not\to x$. Define $(x_n) = f^{-1}(y_n)$. Since $(x_n) \not\to x$ There exists a $\delta > 0$ such that

$$|\{n : (x_n) \notin (x - \delta, x + \delta)\}| = \infty$$

. Since $A/(x - \delta, x + \delta)$ is compact there exists a subsequence $(x_{n_k}) \subset A/(x - \delta, x + \delta)$ such that $(x_{n_k}) \to x_0$ where $x_0 \neq x$. Because f is continuous

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(f^{-1}(y_n)) = \lim_{n\to\infty} y_n = f(x)$$

. This contradicts the assumption that f is one-to-one. Therefore f^{-1} is continuous.

3 5.2.6 (b)Problem Statement

Assume A is open. If g is differentiable at $c \in A$ show

$$g`(c) = \lim_{h\to 0} \frac{g(c+h) - g(c-h)}{2h}$$

3.1 Solution

By 5.2.6 (a) $g`(c) = \lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$.
Let $(x_n) \to 0$, where $x_n \neq 0$ for all values of n. Let (y_n) be defined by $y_n = -x_n$, then $(y_n) \to 0$ and $y_n \neq 0$. By the definition of a limit, $\lim_{n\to\infty} \frac{f(c+y_n)-f(c)}{y_n} = g`(c)$.
We also know that $\lim_{n\to\infty} \frac{f(c+y_n)-f(c)}{y_n} = \lim_{n\to\infty} \frac{f(c-x_n)-f(c)}{-x_n}$.
Therefore

$$\lim_{n\to\infty} \frac{f(c-x_n) - f(c)}{-x_n} = \lim_{n\to\infty} \frac{f(c) - f(c-x_n)}{x_n} = g`(c)$$

. Since our choice of (x_n) was arbitrary, we have that

$$\lim_{h\to 0} \frac{g(c) - g(c-h)}{h} = g`(c)$$

. Putting this together we have

$$g`(c) = \frac{1}{2}(g`(c)+g`(c)) = \frac{1}{2}\left(\lim_{h\to 0} \frac{g(c+h) - g(c)}{h} + \lim_{h\to 0} \frac{g(c) - g(c-h)}{h}\right) = \lim_{h\to 0} \frac{g(c+h) - g(c-h)}{2h}$$

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