MATH 3162 Homework Assignment 2 Solutions

4.4.13(a): Assume that $f : A \to \mathbb{R}$ is uniformly continuous on A and that (x_n) is a Cauchy sequence in A. Choose any $\epsilon > 0$. By definition of uniform continuity, there exists $\delta > 0$ so that for any $a, b \in A$ with $|a - b| < \delta$, it is true that $|f(a) - f(b)| < \epsilon$. Since (x_n) is Cauchy, there exists N so that for $n, m > N$, $|x_n - x_m| < \delta$. Therefore, $|f(x_n) - f(x_m)| < \epsilon$. But we have then shown that for any $\epsilon > 0$, there exists N so that $n, m > N$ implies $|f(x_n) - f(x_m)| < \epsilon$, and so by definition $(f(x_n))$ is Cauchy.

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4.5.2(a): There are many examples of such objects. For instance, define $f(x) = \sin x$ on $(0, 2\pi)$. Clearly f is continuous on $(0, 2\pi)$, and $f((0, 2\pi)) = [-1, 1]$.

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4.5.2(b): This is impossible; any closed interval is compact, and so $f([a, b])$ must be compact, but no open interval is compact.

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4.5.2(c): There are many examples. For instance, define $f(x) = \frac{1}{1-x^2}$ on $(-1, 1)$. Clearly f is continuous on $(-1, 1)$, and $f(-1, 1) = [1, \infty)$, which is closed and unbounded, but not $\mathbb{R}$.

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4.5.3: Assume that f is increasing on $[a, b]$ and satisfies the Intermediate Value Property. Choose any $c \in [a, b]$; we want to show that f is continuous at c. Start with the case $c \in (a, b)$, and choose any $\epsilon > 0$. We claim that we can find m, M so that $m < c < M$ and $f(m) > f(c) - \epsilon$, $f(M) < f(c) + \epsilon$. If $f(a) \geq f(c) - 0.5\epsilon$, then just define $m = a$. If $f(a) < f(c) - 0.5\epsilon$, then we can use the Intermediate Value Theorem (with $L = f(c) - 0.5\epsilon$) to find $m \in (a, c)$ for which $f(m) = f(c) - 0.5\epsilon > f(c) - \epsilon$. Similarly, if $f(b) \leq f(c) + 0.5\epsilon$, then just define $M = b$. If $f(b) > f(c) + 0.5\epsilon$, then we can use the Intermediate Value Theorem (with $L = f(c) + 0.5\epsilon$) to find $M \in (c, b)$ for which $f(M) = f(c) + 0.5\epsilon < f(c) + \epsilon$. Now, just define $\delta = \min(c - m, M - c)$. If $|x - c| < \delta$, then $x \in (c - \delta, c + \delta) \subset (m, M)$, and so since f is increasing, $f(m) \leq f(x) \leq f(M)$, meaning that $f(c) - \epsilon < f(x) < f(c) + \epsilon \implies |f(x) - f(c)| < \epsilon$. Since ϵ was arbitrary, we have verified continuity of f at c.

We still have to deal with the cases $c = a$ or $c = b$. If $c = a$, choose any $\epsilon > 0$. We proceed just as above to find $M > a$ so that $f(M) < f(a) + \epsilon$. Then, just define $\delta = M - a$. If $|x - a| < \delta$, then $x \in (a - \delta, a + \delta)$, and since f is defined only on $[a, b]$, in fact $a \leq x < M$. Then, since f is increasing, $f(a) \leq f(x) \leq f(M) \implies f(a) \leq f(x) < f(a) + \epsilon \implies |f(x) - f(a)| < \epsilon$, and again we verified continuity of f at a. The proof for $c = b$ is almost identical.

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5.2.6(a): By definition, we know that $g'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$. We claim that also $g'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$. To see this, take any sequence $h_n \to 0$ where $h_n \neq 0$ for all n. Then define $x_n = c + h_n$; clearly $x_n \to c$ and $x_n \neq c$ for all c. Therefore, by definition of limit, $\frac{f(x_n)-f(c)}{x_n-c} \to g'(c)$. But $\frac{f(x_n)-f(c)}{x_n-c} = \frac{f(c_n+h)-f(c)}{h}$, and so we know that $\frac{f(c_n+h)-f(c)}{h} \to g'(c)$. Since (h_n) was arbitrary, we've shown that $g'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$.

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Extra problem 1: Assume that f is continuous and 1-1 on $[a, b]$. Since f is 1-1, either $f(a) < f(b)$ or $f(a) > f(b)$. Let's start with the case where $f(a) < f(b)$. We wish to prove that in this case, f is strictly increasing on $[a, b]$. Choose an arbitrary pair x, y with $x < y$, and assume for a contradiction that $f(x) \geq f(y)$. Again, since f is 1-1, $f(x) \neq f(y)$, and so $f(x) > f(y)$. We claim that either $f(a) < f(y)$ or $f(x) < f(b)$. Indeed, if neither of these were true, then $f(y) \leq f(a) < f(b) \leq f(x)$, meaning that $f(y) < f(x)$, which is false. We now have two cases.

Case 1: $f(a) < f(y)$. Then $f(a) < f(y) < f(x)$ and $a < x < y$ (note that $a \neq x$ since $f(a) \neq f(x)$). So, by the Intermediate Value Theorem (with $L = f(y)$), there exists $c \in (a, x)$ so that $f(c) = f(y)$. However, this contradicts the fact that f is 1-1; $c < x < y$, so $c \neq y$, but $f(c) = f(y)$.

Case 2: $f(x) < f(b)$. Then $f(y) < f(x) < f(b)$ and $x < y < b$ (note that $y \neq b$ since $f(y) \neq f(b)$). So, by the Intermediate Value Theorem (with $L = f(x)$), there exists $c \in (y, b)$ so that $f(c) = f(x)$. However, this contradicts the fact that f is 1-1; $x < y < c$, so $c \neq x$, but $f(c) = f(x)$.

In each case we have a contradiction, so our original assumption was wrong, i.e. $f(x) < f(y)$. Since $x < y$ in $[a, b]$ were arbitrary, we've shown that f is strictly increasing on $[a, b]$ when $f(a) < f(b)$. The proof that f is strictly decreasing when $f(a) > f(b)$ is almost identical. (Or, you could replace f with $-f$ to say that this fact follows without loss of generality!)

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Extra problem 2: Assume that f is continuous on $[a, b]$, $f(a) < 0 < f(b)$, and define $S = \{x : f(x) \leq 0\}$. Define $c = \sup S$. Assume for a contradiction that $f(c) < 0$. Then we define $\epsilon = -f(c)$ (remember that $f(c)$ is negative, so $\epsilon > 0$), and by definition of continuity, there exists $\delta > 0$ so that for every $x \in (c - \delta, c + \delta)$, $f(x) \in (f(c) - \epsilon, f(c) + \epsilon) = (2f(c), 0)$. In particular, this means that $f(c + 0.5\delta) < 0$. However, then by definition $c + 0.5\delta \in S$, which means that c is not an upper bound of S, a contradiction to definition of $\sup S$. Therefore, our original assumption was wrong, and $f(c)$ is not negative.

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Extra problem 3: Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and that $f'(c) > 0$. Then by definition of derivative, if we define $d(x) = \frac{f(x)-f(c)}{x-c}$, then $\lim_{x \to c} d(x) = f'(c)$. Choose $\epsilon = f'(c)$. Then by definition of convergence, there exists $\delta > 0$ so that for every $x \in (c-\delta, c+\delta)$, $d(x) \in (f'(c) - \epsilon, f'(c) + \epsilon) = (0, 2f'(c))$. In particular, $d(c - 0.5\delta) > 0$ and $d(c + 0.5\delta) > 0$. Then,

$$d(c - 0.5\delta) = \frac{f(c - 0.5\delta) - f(c)}{c - 0.5\delta - c} = \frac{f(c - 0.5\delta) - f(c)}{-0.5\delta} > 0,$$

which means that $f(c - 0.5\delta) - f(c) < 0 \Longrightarrow f(c) > f(c - 0.5\delta)$. Therefore, $f(c)$ is not a minimum value of $f(x)$ on $\mathbb{R}$. Similarly,

$$d(c + 0.5\delta) = \frac{f(c + 0.5\delta) - f(c)}{c + 0.5\delta - c} = \frac{f(c + 0.5\delta) - f(c)}{0.5\delta} > 0,$$

which means that $f(c + 0.5\delta) - f(c) > 0 \Longrightarrow f(c) < f(c - 0.5\delta)$. Therefore, $f(c)$ is not a maximum value of $f(x)$ on $\mathbb{R}$.

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Extra problem 4: (a) Assume that $f'(x) \neq 0$ for all $x \in [a, b]$. Then there cannot be values $y, z \in [a, b]$ with 0 between $f'(y)$ and $f'(z)$; otherwise, by Darboux's Theorem, there would exist c between y and z with $f'(c) = 0$, a contradiction. So, either $f'(x) > 0$ for all $x \in [a, b]$ or $f'(x) < 0$ for all $x \in [a, b]$.

(b) By part (a), either $f'(x) > 0$ for all $x \in [a, b]$ or $f'(x) < 0$ for all $x \in [a, b]$. Assume that $f'(x) > 0$ for all $x \in [a, b]$. Then, for every $y < z \in [a, b]$, by the Mean Value Theorem, there exists $c \in [y, z]$ with $f'(c) = \frac{f(z)-f(y)}{z-y}$. Since $f'(c) > 0$ by assumption, and $z - y > 0$, it must be true that $f(z) - f(y) > 0$, i.e. $f(y) < f(z)$. Since $y < z$ were arbitrary, this shows that f is strictly increasing on $[a, b]$. If instead it were the case that $f'(x) < 0$ for all $x \in [a, b]$, then virtually the same proof shows that f is strictly decreasing on $[a, b]$.

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