MATH 3162 Homework Assignment 3 Solutions

5.2.9(a): Suppose that f' exists on $[a, b]$ and is not constant. Then there exist $x < y \in [a, b]$ so that $f'(x) \neq f'(y)$. Since $\mathbb{Q}^c$ is dense, there exists an irrational number α between $f'(x)$ and $f'(y)$, and so by Darboux's Theorem, there exists $c \in (x, y)$ so that $f'(c) = \alpha$. Therefore, f' indeed must achieve an irrational value.

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5.3.7: Assume for a contradiction that f is differentiable on an interval (a, b), $f'(x) \neq 1$ on (a, b), and f has two fixed points there, i.e. there exist $x < y$ in (a, b) for which $f(x) = x$ and $f(y) = y$. Then, f is differentiable on $[x, y]$, and so by the Mean Value Theorem there exists $c \in (x, y)$ so that $f'(c) = \frac{f(y)-f(x)}{y-x} = \frac{y-x}{y-x} = 1$. This is a contradiction to the assumption that f' never equals 1, so our original assumption was wrong and so f has at most one fixed point on (a, b).

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5.3.8: Choose any sequence (x_n) approaching 0 where $x_n > 0$ for all n. For every n, f is differentiable on $(0, x_n)$, and so by the Mean Value Theorem, there exists $c_n \in (0, x_n)$ so that $f'(c_n) = \frac{f(x_n)-f(0)}{x_n-0}$. Since $0 < c_n < x_n$ and $x_n \to 0$, $c_n \to 0$ as well, and so $f'(c_n)$ approaches L since we know that $\lim_{x\to 0} f'(x) = L$. But this implies that $\frac{f(x_n)-f(0)}{x_n-0}$ approaches L. Since (x_n) was arbitrary, we've shown by definition that

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = L.$$

A nearly identical proof shows that

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = L.$$

Therefore,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = L,$$

implying that $f'(0) = L$.

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5.4.5: By definition of g,

$$g(x_m) = g(2^{-m}) = h(2^{-m}) + 2^{-1}h(2^{-m+1}) + 2^{-2}h(2^{-m+2}) + \ldots + 2^{-m}h(1) + 2^{-m-1}h(2) + \ldots$$

Since $h(x) = 2$ for every even integer x and $h(x) = x$ for $0 < x < 1$, we can rewrite as

$$g(x_m) = g(2^{-m}) = 2^{-m} + 2^{-1}2^{-m+1} + 2^{-2}2^{-m+2} + \ldots + 2^{-m}1 = 2^{-m} + 2^{-m} + \ldots + 2^{-m} = (m+1)2^{-m}.$$

Therefore,

$$\frac{g(x_m) - g(0)}{x_m - 0} = \frac{(m+1)2^{-m} - 0}{2^{-m} - 0} = m + 1.$$

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6.2.1(a): For every $x \neq 0$, we claim that $f_n(x) \to \frac{1}{x}$. To see this, note that we can rewrite

$$f_n(x) = \frac{x}{1/n + x^2}.$$

Since $1/n \to 0$ as $n \to \infty$, $f_n(x) \to \frac{x}{x^2} = \frac{1}{x}$ since $x \neq 0$. On the other hand, if $x = 0$, then $f_n(x) = f_n(0) = \frac{0}{1} = 0$ for all n, and so $f_n(0) \to 0$.

Therefore, (f_n) approaches the limit f pointwise, where f is defined by $f(0) = 0$ and $f(x) = \frac{1}{x}$ for $x \neq 0$.

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6.2.1(b): No, it is not. To see this, fix $\epsilon = 1$, and we will demonstrate, for every $N \in \mathbb{N}$, examples of $x \in (0, \infty)$ and $n > N$ so that $|f_n(x) - f(x)| \geq \epsilon$.

Specifically, for every N, define $n = N+1$ and $x = \frac{1}{N+1}$. Then $f(x) = N+1$, and $f_n(x) = \frac{(N+1)x}{1+(N+1)x^2} = \frac{1}{1+\frac{1}{N+1}} = \frac{N+1}{N+2} < 1$. Therefore, $|f_n(x) - f(x)| \geq N \geq 1 = \epsilon$, completing the proof of the negation of uniform convergence.

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6.2.1(d): Yes, it is. To see this, write

$$|f(x) - f_n(x)| = \left|\frac{1}{x} - \frac{nx}{1+nx^2}\right| = \left|\frac{1+nx^2-nx^2}{x(1+nx^2)}\right| = \frac{1}{x(1+nx^2)}.$$

For every $x > 1$, $x(1+nx^2) > n$, and so for every $x \in (1, \infty)$, $|f(x) - f_n(x)| < \frac{1}{n}$. Then, for every $\epsilon > 0$, you can choose N for which $\frac{1}{N} < \epsilon$, and then for every $n > N$ and every $x \in (1, \infty)$, $|f(x) - f_n(x)| < \frac{1}{n} < \frac{1}{N} < \epsilon$, completing the proof of uniform convergence.

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6.2.7: Suppose that f is uniformly continuous on $\mathbb{R}$. Choose any $\epsilon > 0$, and choose δ so that for all $x, y \in \mathbb{R}$, $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Then, by the Archimedean Principle, there exists N so that $\frac{1}{N} < \delta$. Then, for every $x \in \mathbb{R}$ and every $n > N$, $|(x + 1/n) - x| = 1/n < \frac{1}{N} < \delta$, and so by definition of δ, $|f(x + 1/n) - f(x)| < \epsilon$. But this means that $|f_n(x) - f(x)| < \epsilon$, and since $n > N$ and x was arbitrary, this proves that $f_n \to f$ uniformly.

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6.2.8: Suppose that g_n are all continuous on a compact set K, $g_n \to g$ uniformly on K, and $g(x) \neq 0$ for all $x \in K$. First, g is the uniform limit of continuous functions, and so g is continuous on K. If $g(x)$ could be both positive and negative for values $x \in K$, then by the Intermediate Value Theorem, there would exist c where $g(c) = 0$, a contradiction. Therefore, g is either positive for all $x \in K$ or negative for all $x \in K$. We'll first treat the first case, i.e. $g(x) > 0$ for all $x \in K$. Since K is compact, g achieves a minimum on K, i.e. there exists c so that $g(c) \leq g(x)$ for all $x \in K$. We'll use M to denote $g(c)$; note that $M > 0$.

Choose any $\epsilon > 0$. Apply uniform convergence with $\frac{M}{2}$. This yields N_1 so that for every $n > N_1$ and $x \in K$, $|g_n(x) - g(x)| < \frac{M}{2}$. For any such n and x, $|g_n(x)| \geq |g(x)| - |g(x) - g_n(x)| > M - \frac{M}{2} = \frac{M}{2}$ by the triangle inequality. Now, again by uniform convergence, choose N_2 so that for every $n > N_2$ and $x \in K$, $|g_n(x) - g(x)| < \frac{\epsilon M^2}{2}$, and define $N = \max(N_1, N_2)$. Then, for any $n > N$, clearly $n > N_1$ and $n > N_2$. Then, for any $x \in K$,

$$\left| \frac{1}{g(x)} - \frac{1}{g_n(x)} \right| = \frac{|g_n(x) - g(x)|}{|g(x)||g_n(x)|} < \frac{\epsilon M^2/2}{M \cdot (M/2)} = \epsilon$$

since $|g_n(x) - g(x)| < \frac{\epsilon M^2}{2}$ (using $n > N_2$), $|g(x)| \geq M > 0$ (using the fact that $M = f(c)$ is the minimum of $f(x)$ on K), and $|g_n(x)| \geq \frac{M}{2} > 0$ (using $n > N_1$). Since $x \in K$ and $n > N$ were arbitrary, $\frac{1}{g_n(x)} \to \frac{1}{g(x)}$ uniformly.

If instead g is negative on all of K, then a similar argument can be used with M equal to the negative of the maximum of g on K.

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