Math 3162 Homework Assignment 6 Grad Problem Solutions

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1 Problem Statement 7.2.7

Let $f : [a, b] \to \mathbb{R}$ be increasing on the set $[a, b]$. Show that f is integrable on $[a, b]$.

1.1 Solution

This problem is a great introduction to a telescoping series. Although I did not construct a telescoping series in this proof, I could have. The telescoping behavior is what makes the proof possible, it allows us to cancel almost all terms in the sums of $U(f, P_\epsilon)$ and $L(f, P_\epsilon)$.
Proof:
Since f is increasing on $[a, b]$, f is bounded by $M = Max\{|f(a)|, |f(b)|\}$. Therefore, we may use the Integrability Criterion of Thm 7.2.8. Let $\epsilon > 0$.
Let $n = \lceil \frac{(f(b)-f(a))(b-a)}{\epsilon} \rceil$
Let $P_\epsilon = \{a + \frac{i}{n}(b - a) : 0 < i < n\}$. Notice that any two neighboring points in the partition are of distance $\frac{(b-a)}{n}$.
Then because f is increasing and all intervals in the partition are of equal length, the $\sup$ of $f(x)$ over any interval is achieved at the right endpoint.
We have

$$U(f, P_\epsilon) = \sum_{i=0}^{n-1} f(a + \frac{i+1}{n}(b - a))\frac{(b - a)}{n} =$$

$$\frac{(b - a)}{n}\left\{f(a + \frac{n}{n}(b - a)) + \sum_{i=0}^{n-2} f(a + \frac{i+1}{n-1}(b - a))\right\} =$$

$$\frac{(b - a)}{n}\left\{f(b) + \sum_{i=0}^{n-2} f(a + \frac{i+1}{n-1}(b - a))\right\}.$$

Similarly, we have that the $\inf$ of $f(x)$ over any interval is achieved at the left endpoint.
So

$$L(f, P_\epsilon) = \sum_{i=0}^{n-1} f(a + \frac{i}{n}(b - a))\frac{(b - a)}{n} =$$

$$\frac{(b - a)}{n}\left\{f(a + \frac{0}{n}(b - a)) + \sum_{i=1}^{n-1} f(a + \frac{i}{n}(b - a))\right\} =$$

$$\frac{(b-a)}{n}\left\{f(a)+\sum_{i=1}^{n-1} f(a+\frac{i}{n}(b-a))\right\} =$$

$$\frac{(b-a)}{n}\left\{f(a)+\sum_{i=0}^{n-2} f(a+\frac{i+1}{n}(b-a))\right\}.$$

So

$$U(f,P_\epsilon)-L(f,P_\epsilon)=$$

$$\frac{(b-a)}{n}\left\{f(b)-f(a)+\sum_{i=0}^{n-2} f(a+\frac{i+1}{n-1}(b-a))-\sum_{i=0}^{n-2} f(a+\frac{i+1}{n}(b-a))\right\} =$$

$$\frac{(b-a)(f(b)-f(a))}{n} < (b-a)(f(b)-f(a))\frac{\epsilon}{(b-a)(f(b)-f(a))} \leq \epsilon.$$

Since our choice of ϵ was arbitrary, by Thm 7.2.8 f is integrable on $[a,b]$.