MATH 3162 Homework Assignment 6 Solutions

6.6.4: Consider $f(x) = \ln(1+x)$. We claim that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ is the Taylor series of $f(x)$, and we will prove that it converges at $x = 1$. First, we check that the coefficients are correct. $f'(x) = (1+x)^{-1}$, $f''(x) = -1!(1+x)^{-2}$, $f'''(x) = 2!(1+x)^{-3}$, $f^{(4)}(x) = -3!(1+x)^{-4}$, etc. It is easy to show by induction that $f^{(n)}(x) = (-1)^{n+1}(n-1)!(1+x)^{-n}$. Therefore, for the Taylor series, $a_0 = f(0) = \ln 1 = 0$, and for $n > 1$, $a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$ as desired. Therefore, indeed $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ is the Taylor series of $f(x)$.

Now, we use the Lagrange Remainder Theorem at $x = 1$ to prove convergence there. By that theorem, for every N, there exists $c \in (0,1)$ s.t.

$$|E_N(1)| = \frac{|f^{(N+1)}(c)|}{(N+1)!} 1^n = \frac{N!(1+c)^{-(N+1)}}{(N+1)!} = \frac{1}{(N+1)(1+c)^{N+1}}.$$

Since $c > 0$, this shows that $|E_N(1)| < \frac{1}{N+1}$. Since $\frac{1}{N+1} \to 0$, this implies that $E_N(1) \to 0$ (as $N \to \infty$), and therefore that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ converges to $\ln(1+x)$ at $x = 1$. Therefore,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots = \ln 2.$$

■

6.7.2: Choose any $a < b$, any continuous f on $[a,b]$, and any $\epsilon > 0$. Since f is continuous on the compact set $[a,b]$, it is uniformly continuous. Therefore, there exists $\delta > 0$ so that for $x, y \in [a,b]$, $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$. Now, by the Archimedean Principle, there exists n so that $\delta < \frac{b-a}{n}$. Define $x_0 = a$, $x_1 = a + \frac{b-a}{n}$, $x_2 = a + 2\frac{b-a}{n}$, $\ldots$, $x_n = b$; clearly $x_{k+1} - x_k = \frac{b-a}{n} < \delta$ for all k. Finally, define a polygonal function ϕ by connecting the points $(x_k, f(x_k))$ for $0 \le k \le n$, i.e. for every k, $\phi(x)$ is defined on the interval $[x_k, x_{k+1}]$ to be the line segment between $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$.

We claim that for every $x \in [a,b]$, $|f(x) - \phi(x)| < \epsilon$. To see this, choose any $x \in [a,b]$. There must exist k so that $x_k \le x \le x_{k+1}$. Since ϕ is a line segment between $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$ on the interval $[x_k, x_{k+1}]$, $\phi(x)$ is between $f(x_k)$ and $f(x_{k+1})$. Since f is continuous on $[x_k, x_{k+1}]$, by the IVT there exists $y \in [x_k, x_{k+1}]$ so that $f(y) = \phi(x)$. Then, since $x, y \in [x_k, x_{k+1}]$, $|x - y| \le x_{k+1} - x_k < \delta$. Therefore, $|f(x) - \phi(x)| = |f(x) - f(y)| < \epsilon$. Since x was arbitrary, this completes the proof.

■

6.7.9(a): Define $f(x) = \frac{1}{x}$. Then f is continuous on $(0,1)$. However, there is no way for a sequence of polynomials p_n to uniformly approach f on $(0,1)$; if this were the case, then for $\epsilon = 1$, there would exist n so that $|p_n(x) - 1/x| < 1$ for all $x \in (0,1)$, which would imply that $|1/x| \le |p_n(x)| + |(1/x) - p_n(x)| < |p_n(x)| + 1$

for all $x \in (0,1)$. However, this is clearly impossible; p_n is continuous on $[0,1]$ and therefore bounded on $[0,1]$, and so also bounded on $(0,1)$, whereas $1/x$ is clearly unbounded on $(0,1)$.

■

7.2.4: If $L(g,P) = U(g,P)$, then

$$0 = U(g,P) - L(g,P) = \left(\sum_{k=0}^{n-1} M_k(x_{k+1} - x_k)\right) - \left(\sum_{k=0}^{n-1} m_k(x_{k+1} - x_k)\right) = \sum_{k=0}^{n-1}(M_k - m_k)(x_{k+1} - x_k).$$

Since each $x_{k+1} - x_k$ is positive and each $M_k - m_k \geq 0$, this implies that every $M_k - m_k = 0$, i.e. that $m_k = M_k$ for each k. This means that for every k, $\sup\{g(x) : x \in [x_k, x_{k+1}]\} = \inf\{g(x) : x \in [x_k, x_{k+1}]\}$, which implies that $\{g(x) : x \in [x_k, x_{k+1}]\}$ is a singleton set. In other words, for every k, g is constant on $[x_k, x_{k+1}]$, let's say that $g(x) = a_k$ for all $x \in [x_k, x_{k+1}]$. Note that $a_0 = g(x_1) = a_1$, and so $a_0 = a_1$. Similarly, $a_1 = g(x_2) = a_2$, so $a_1 = a_2$. Continuing in this way, all a_k are equal, and so g in fact must be a constant function, say $g(x) = t$ for all $x \in [a,b]$. Therefore, g is obviously continuous and so integrable. For every partition P, $U(g,P) = L(g,P) = \sum_{k=0}^{n-1} t(x_{k+1} - x_k) = t(b-a)$, and so $\int_a^b g(x)\ dx = t(b-a)$.

■

Written problem: Choose any $a < b$, any continuous f on $[a,b]$, and any $\epsilon > 0$. Define $M = \max(|a|,|b|)$. We may assume without loss of generality that $M \geq 1$; if this were not the case, then we could make it true by making the interval $[a,b]$ larger, and if we prove the desired conclusion on a larger interval, it clearly holds on the original as well. By the Weierstrass Approximation Theorem, there exists a polynomial $p(x)$ so that $|f(x) - p(x)| < \epsilon/2$ for all $x \in [a,b]$. Write $p(x) = a_0 + a_1 x + \ldots + a_n x^n$. Then, since $\mathbb{Q}$ is dense, for each i from 0 to n we can find a rational r_i satisfying $|a_i - r_i| < \frac{\epsilon}{2M^n(n+1)}$. Then, define the polynomial $q(x) = r_0 + r_1 x + \ldots + r_n x^n$. For every $x \in [a,b]$, $|x| \leq M$, and so

$$\begin{aligned}
|p(x) - q(x)| &= |(a_0 - r_0) + (a_1 - r_1)x + \ldots + (a_n - r_n)x^n| \\
&\leq |a_0 - r_0| + |a_1 - r_1||x| + \ldots + |a_n - r_n||x|^n \\
&< \frac{\epsilon}{2M^n(n+1)} + \frac{\epsilon}{2M^n(n+1)}M + \ldots + \frac{\epsilon}{2M^n(n+1)}M^n \\
&\leq \frac{\epsilon}{2M^n(n+1)}M^n + \frac{\epsilon}{2M^n(n+1)}M^n + \ldots + \frac{\epsilon}{2M^n(n+1)}M^n = \frac{\epsilon}{2}.
\end{aligned}$$

Finally, this shows that for every $x \in [a,b]$, $|f(x) - q(x)| = |f(x) - p(x) + p(x) - q(x)| \leq |f(x) - p(x)| + |p(x) - q(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

■